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The turbulent convective flux of the toroidal angular momentum density is derived using the nonlinear toroidal gyrokinetic equation which conserves phase space density and energy [T. S. Hahm, Phys. Fluids, **31**, 2670 (1988)]. A novel pinch mechanism is identified which originates from the symmetry breaking due to the magnetic field curvature. A net parallel momentum transfer from the waves to the ion guiding centers is possible when the fluctuation intensity varies on the flux surface, resulting in imperfect cancellation of the curvature drift contribution to the parallel acceleration. This mechanism is inherently a toroidal effect, and complements the $k_\parallel$ symmetry breaking mechanism due to the mean $\mathbf{E}\times\mathbf{B}$ shear [O. Gurcan *et al.*, Phys. Plasmas **14**, 042306 (2007)] which exists in a simpler geometry. In the absence of ion thermal effects, this pinch velocity of the angular momentum density can also be understood as a manifestation of a tendency to homogenize the profile of "magnetically weighted angular momentum density," $nm_iR^2\omega_\parallel/B^2$. This part of the pinch flux is mode-independent (whether it is trapped electron mode or ion temperature gradient mode driven), and radially inward for fluctuations peaked at the low-B-field side, with a pinch velocity typically, $V_{\rm Ang}^{\rm TEP}\sim -2\chi_\phi/R_0$. Ion thermal effects introduce an additional radial pinch flux from the coupling with the curvature and grad-B drifts. This curvature driven thermal pinch can be inward or outward, depending on the mode-propagation direction. Explicit formulas in general toroidal geometry are presented. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2743642]

I. INTRODUCTION

It is well known that plasma rotation can play a crucial role in reducing turbulence and transport as well as in stabilizing magnetohydrodynamic (MHD) instabilities including the resistive wall mode (RWM). Therefore, understanding momentum transport which influences the plasma rotation is a very important issue. However, current theoretical understanding of the momentum transport lags behind that of ion thermal transport, if not that of the electron thermal transport and particle transport.

Transport analysis of tokamak experiments usually indicates that the toroidal momentum diffusivity χ_ϕ is anomalous, i.e., higher than neoclassical theory predictions from collisional transport mechanisms. Typically, χ_ϕ is comparable to the ion thermal diffusivity χ_i,[1] in rough agreement with theoretical predictions based on low frequency, ion gyroradius scale, electrostatic drift wave turbulence, including ion temperature gradient (ITG) mode turbulence[2] and trapped electron mode (TEM) turbulence.[3] However, the observation of spontaneous toroidal rotation of plasmas in the absence of apparent torque input brought new challenges for theoretical understanding. Spontaneous rotation has been observed in many tokamaks.[4–10] In particular, it has been explored in detail by the Alcator C-Mod team and others[5–7,9–11] and is sometimes called an "intrinsic rotation."[7] The variety of rotation behavior in many tokamaks seems to indicate that it is not possible to explain most rotation profiles, which are sometimes peaked near the axis where there is no torque input, using an "anomalous diffusion" of momentum only. A likely dynamical scenario for the origin of spontaneous rotation involves a nondiffusive inward flux of toroidal angular momentum from edge sources. In addition, a recent perturbation experiment on JT60-U neutral beam heated plasmas showed a need for an "inward pinch term" of angular momentum in the transient transport analysis, to match the measured centrally peaked rotation profiles.[12,13]

Recognizing a need for theoretical identification of a pinch mechanism (or to be more generic, a nondiffusive component of the radial transport of toroidal momentum[14]), there has been renewed interest in establishing physical mechanisms for nondiffusive momentum transport. These include recent work by Gurcan *et al.*,[15] where the role of the $\mathbf{E}\times\mathbf{B}$ shear in inducing a nondiffusive component of toroidal momentum transport is elucidated and quantitatively calculated. To obtain a nondiffusive flux of parallel momentum, it is necessary to produce a net acceleration of the ion flow parallel to the equilibrium magnetic field. In nonlocal analysis, this acceleration is proportional to the radial average of $k_\parallel$ over the spectral width, which usually vanishes in a simple analysis, since the eigenmode is peaked at the rational surface and $k_\parallel \propto m - \ell q$ flips sign at the mode rational surface,[14] where ℓ is the toroidal mode number. However, the $\mathbf{E}\times\mathbf{B}$ shear provides a robust symmetry breaking mechanism,[15] which is necessary for net plasma acceleration, by radially shifting the eigenmode to one side, and thereby making the radial average of $k_\parallel$ nonzero. One obtains a similar, but much weaker effect from the parallel velocity shear.[16] A nonzero value of $\langle k_\parallel \rangle$ also implies a finite mean parallel wave momentum, since the wave-momentum density is $\mathbf{P}=\mathbf{k}N$, with N the wave population (action) density. That work exhibited several promising features including the observation of co-rotation of many H-mode plasmas, produced via various methods, in tokamaks[17] in which $\mathbf{E}\times\mathbf{B}$ shear is expected to be significant. In particular, the theory predicts a $\nabla P_i/n_i$ shear driven residual stress (i.e., neither diffusion nor pinch) which, acting in concert with the edge boundary condition on the flow, can drive "intrinsic" rotation. A residual stress-like term may be needed to explain a recent result from TCV.[18]

On the other hand, spontaneous rotation has also been observed in low-mode (L-mode)[5] and Ohmically heated (OH) plasmas[7,11] in which the mean $\mathbf{E}\times\mathbf{B}$ shear effect is expected to be weak. Therefore, it is worthwhile to explore other possible physical mechanisms for an inward pinch of toroidal angular momentum in the absence of mean $\mathbf{E}\times\mathbf{B}$ shear.

In this paper, we develop a general nonlinear expression for the radial flux of the ion parallel angular momentum density using the electrostatic toroidal nonlinear gyrokinetic equations with proper conservation laws, including those of phase-space density and energy.[19] From this study, we identify a novel pinch mechanism for the parallel angular momentum density which originates from the symmetry breaking due to the equilibrium B field curvature and inhomogeneity. In this analysis, $E_r'=0$ throughout. It is expected that turbulence-driven $\mathbf{E}\times\mathbf{B}$ zonal flows[20] exist in OH and L-mode plasmas. However, unlike mean $\mathbf{E}\times\mathbf{B}$ flow shear, the zonal flow shear has no preferred sign in a statistical sense. Therefore, there will be no direct $k_\parallel$ symmetry breaking due to turbulence driven zonal flows. Throughout this paper, we ignore the effect of turbulence driven zonal flows.

From our work, the resulting radial component of the turbulence driven flux, $\mathbf{\Pi}_{\rm Ang}$, of the ion parallel momentum density $m_i n_0 U_\parallel R$ can be written as

$$\langle \mathbf{\Pi}_{\rm Ang}^{\rm Turb} \cdot \nabla\psi \rangle = -\chi_{\rm Ang}\left\langle (RB_\theta)^2 \frac{\partial}{\partial\psi}(m_i n_0 U_\parallel R)\right\rangle + V_{\rm Ang}\langle RB_\theta m_i n_0 U_\parallel R\rangle,$$

where ψ is the poloidal flux designating the radial coordinate with the relation $d\psi = RB_\theta dr$. In the hydrodynamic limit,

$$\chi_{\rm Ang} \equiv \left\langle \sum_{\mathbf{k}} \mathrm{Re}\ \tau_{c\mathbf{k}} |\delta v_{r\mathbf{k}}|^2 \right\rangle = \left\langle \left(\frac{c}{RB_\theta}\right)^2 \sum_{\mathbf{k}} \mathrm{Re}\ \tau_{c\mathbf{k}} \ell^2 |\delta\phi_{\mathbf{k}}|^2 \right\rangle$$

is the flux-surface-averaged turbulent angular momentum density diffusivity, where Re $\tau_{c\mathbf{k}}$ is the turbulence decorrelation time. The novel turbulence driven convective pinch velocity $V_{\rm Ang}^{\rm TurCo}$ consists of two parts with different physical origins. To the lowest order in r/R_0, with R_0 the major radius at the magnetic axis, the turbulent equipartition pinch velocity, $V_{\rm Ang}^{\rm TEP}$ is driven by $\nabla(1/B)$, and given by

$$V_{\rm Ang}^{\rm TEP} \simeq -\frac{2F_{\rm balloon}}{R_0}\chi_{\rm Ang},$$

where a dimensionless coefficient on the order of unity, $F_{\rm balloon}$ characterizes the "ballooning structure" of the turbulence. This is defined after Eq. (41), in relation to Table II. For typical outward ballooning fluctuations (peaked at the low-B side), $F_{\rm balloon}\sim 1>0$, and $V_{\rm Ang}^{\rm TEP}<0$, i.e., inward in radius. This part of the prediction comes mostly from the geometric properties of the nonlinear gyrokinetic system, and is insensitive to the propagation direction of the underlying microinstabilities. On the other hand, the curvature driven thermal (CTh) flux is given by

$$V_{\rm Ang}^{\rm CTh} \simeq -\frac{4F_{\rm balloon}G^{\rm Th}}{R_0}\chi_{\rm Ang},$$

and is due to the ion thermal effects associated with the ion temperature fluctuations. This piece is characterized by a dimensionless coefficient on the order of unity, $G^{\rm Th}\simeq(\delta T_i/e_i\delta\phi)$. Since this ratio depends on the direction of mode propagation (very roughly ω_{*Ti}/ω), the sign and magnitude of $V_{\rm Ang}^{\rm CTh}$ depend on the details of underlying microturbulence. For fluctuations propagating in the electron diamagnetic direction, $G^{\rm Th}$ is positive definite, making $V_{\rm Ang}^{\rm CTh}$ inward for outward ballooning fluctuations. On the other hand, for fluctuations propagating in the ion diamagnetic direction, $G^{\rm Th}$ can be negative (though a precise determination of sign requires a numerical evaluation as we discuss in the main text), and $V_{\rm Ang}^{\rm CTh}$ can be outward for outward ballooning fluctuations. So unlike $V_{\rm Ang}^{\rm TEP}$, which is inward regardless of microinstability details, $V_{\rm Ang}^{\rm CTh}$ depends on the mode propagation direction and proximity to linear marginality. We also note that, typically $|V_{\rm Ang}^{\rm CTh}/V_{\rm Ang}^{\rm TEP}|\sim T_i/T_e$. Therefore, we predict that for the TEM-dominated turbulence expected for Ohmic and electron-heated plasmas, the total convective pinch velocity $V_{\rm Ang}^{\rm TurCo}\equiv V_{\rm Ang}^{\rm TEP}+V_{\rm Ang}^{\rm CTh}$ is inward. On the other hand, for ITG-dominated turbulence, $V_{\rm Ang}^{\rm CTh}$ can sometimes be outward, while $V_{\rm Ang}^{\rm TEP}$ is always inward. Therefore, the resulting net sign of $V_{\rm Ang}^{\rm TurCo}$ depends on several factors such as T_i/T_e and the proximity to linear marginality, and a general prediction of the pinch velocity direction is not possible.

As discussed in relation to Ref. 15, a net acceleration of the parallel velocity after an average over the mode width is a key to obtaining a nondiffusive radial flux of the parallel momentum. In a sheared slab or in cylindrical geometry with negligible variation of B or of the curvature of B, a necessary symmetry breaking mechanism required for a net acceleration is provided by the $\mathbf{E}\times\mathbf{B}$ shear, as shown in a nonlinear gyrofluid simulation.[21,22] In strongly magnetized plasmas in toroidal geometry, the nonlinear gyrokinetic equation satisfying the relevant conservation laws[19] indicates that the perturbed gyrocenter parallel velocity $v_\parallel^{(1)}\equiv \mathbf{b}\cdot d\mathbf{R}^{(1)}/dt$ obeys

$$m_i \frac{dv_\parallel^{(1)}}{dt} = -\frac{\mathbf{B}^*}{B^*} \cdot e_i \nabla \langle \delta\phi \rangle$$

$$= -\frac{\mathbf{B} + \frac{m_i c}{e_i} \nabla \times v_\parallel \mathbf{b}}{B^*} \cdot e_i \nabla \langle \delta\phi \rangle,$$

since $\mathbf{B}^* \equiv \mathbf{B} + \frac{m_i c}{e_i} \nabla \times v_\parallel \mathbf{b}$. Therefore, the parallel acceleration of gyrocenters in a strongly magnetized plasmas depends not only on the $k_\parallel$ of the fluctuations, but also on the perturbed $\mathbf{E}\times\mathbf{B}$ velocity which couples to the magnetic curvature $\propto \mathbf{b} \times (\mathbf{b}\cdot\nabla)\mathbf{b}$, orthogonal to $\mathbf{b}$. To obtain a net acceleration, we need either a symmetry breaking in the first term ($\propto k_\parallel$), which is discussed in detail in Ref. 15, or a symmetry breaking in the second term, addressed in this paper, which is related to the magnetic field inhomogeneity. For the latter, since the magnetic curvature changes its sign along the B field as one moves from the low B field (bad curvature) side to the high B field (good curvature) side, the fluctuation amplitude must change along the magnetic field to yield a net acceleration. This is why ballooning structure of the fluctuations is required to obtain the momentum pinch term studied in this paper. These two physically different symmetry breaking mechanisms can be viewed as limiting cases of a more general symmetry breaking mechanism which can be dubbed the "$\mathbf{B}^*$-symmetry breaking."

The remainder of this paper is organized as follows: In Sec. II, the physical mechanism of the parallel angular momentum pinch identified in this work is discussed. From the nonlinear gyrokinetic equation, a moment approach leading to the radial flux of parallel angular momentum density in the hydrodynamic limit is presented in Sec. III, and explicit expressions for the angular momentum pinch and the momentum diffusivity are derived. In Sec. IV, we interpret the ∇B-driven inward pinch of parallel angular momentum density in terms of turbulent equipartition (TEP) theory. We also compare and contrast the pinch with the now familiar TEP mechanism for the density pinch.[23–25] In Sec. V, a quasilinear gyrokinetic expression for the radial flux of parallel angular momentum is presented and compared to the moment results. Finally, our results are discussed in relation to experimental observations and the theory of the curvature driven particle pinch in Sec. VI.

II. ORIGIN OF MOMENTUM PINCH IN TOROIDAL GEOMETRY

In this section, we discuss the physical origin of a novel momentum density pinch in toroidal geometry. Further detailed analyses are presented in the forthcoming sections. The purpose of this section is the identification of terms which lead to a momentum density pinch, rather than a systematic derivation thereof. While the angular momentum density is the quantity of primary physical interest in toroidal systems, for simplicity we first discuss the convective pinch of simple momentum density in this section. Since some transport analyses were implemented for the momentum density in the past, it is also useful to point out some quantitative differences originating from geometric effects such as the dependence on $B \propto 1/R$. In Secs. III–VI, we deal with the angular momentum density explicitly. The radial flux of the toroidal momentum density nU_ϕ driven by the electrostatic turbulence can be written as

$$\Gamma_{\rm Mom} \equiv \langle \delta v_r \delta(nU_\phi) \rangle, \tag{1}$$

where δv_r is the radial component of the fluctuating $\mathbf{E}\times\mathbf{B}$ velocity due to turbulence, and $\delta(nU_\phi)$ is the momentum density fluctuation. Here, $\langle \cdots \rangle$ represents the flux surface average. We will use $\langle\langle \cdots \rangle\rangle$ for the gyrophase average. We note that, since $\delta(nU_\phi) = n_0 \delta U_\phi + U_0 \delta n + \delta n \delta U_\phi$, not only the velocity fluctuations, but also the density fluctuations can contribute to the radial flux of momentum density, since each particle carries its own momentum. Hence, there are both convection ($\sim \langle \delta v_r \delta n \rangle$) and Reynolds stress ($\sim \langle \delta v_r \delta U_\phi \rangle$) contributions to the total momentum flux. There also exists a triplet term $\langle \delta n \delta U_\phi \delta v_r \rangle$ which is a higher order effect which we do not address in this paper. However, triplet terms like this have been shown[26] to be responsible for turbulence spreading,[27–36] which is another outstanding theoretical issue.

In tokamaks where $B_\phi \gg B_\theta$, the "magnitude" of U_ϕ can be approximated by $U_\parallel$, since

$$U_\parallel = \mathbf{U}\cdot\mathbf{b} = (U_\phi \hat{\mathbf{e}}_\phi + U_\theta \hat{\mathbf{e}}_\theta)\cdot(B_\phi \hat{\mathbf{e}}_\phi + B_\theta \hat{\mathbf{e}}_\theta)/B$$

$$= U_\phi \frac{B_\phi}{B} + U_\theta \frac{B_\theta}{B} \simeq U_\phi,$$

if $U_\theta B_\theta / B \ll U_\phi$. Since $k_\parallel \ll k_\perp$, the effect of $\mathbf{U}_\parallel$ on turbulence is relatively weak compared to that of the $\mathbf{E}\times\mathbf{B}$ flow which is perpendicular to $\mathbf{B}$.[37] As is well known from nonlinear theory[37,38] and from experiments,[39,40] $\mathbf{E}\times\mathbf{B}$ shear plays an essential role in reducing turbulence. In this paper, we focus our studies on the radial transport of U_ϕ, rather than on its effect on turbulence. With this in mind, the radial flux of the toroidal momentum is approximated by that of the parallel momentum, and we have

$$\Gamma_{\rm Mom} \equiv \langle \delta(nU_\phi)\delta v_r \rangle \simeq \langle \delta(nU_\parallel)\delta v_r \rangle$$

$$= U_0 \Gamma_{ptl} + n_0 \Pi_{\parallel,r}, \tag{2}$$

where $\Gamma_{\rm ptl} \equiv \langle \delta n \delta v_r \rangle$ is the particle flux (assuming $\delta n_i = \delta n_e$), $\pi_{\parallel,r} \equiv \langle \delta U_\parallel \delta v_r \rangle$ is the parallel Reynolds stress, which has been measured from experiments,[41] and U_0 is a simpler notation for $U_{0,\parallel}$. Therefore, in discussing momentum transport, contributions from particle transport should be kept in mind. For instance, particle flux can manifest itself as part of an apparent momentum pinch, if one considers the flux of $U_\parallel$. As will become more apparent in the forthcoming sections, a formulation in terms of the (angular) momentum density (rather than in terms of momentum or velocity $U_\parallel$) is most natural. We also note that calculating the turbulent particle flux from the ion response alone can be misleading. This is because of the quasineutrality constraint on the density response. Indeed the expression $\Gamma_{\rm ptl}$ is merely an apparent, test-particle-type radial flux of ion guiding centers. Given the subtlety of all these interconnections between momentum, angular momentum, and particle transport, we defer any further discussion of particle flux coupling to Appendix A.

In this paper, we show that a careful treatment of geometric effects due to nonuniform $\mathbf{B}$ yields a novel pinch mechanism for parallel (angular) momentum density. Before presenting more detailed systematic derivations in Secs. III and V, here we discuss the basic physics mechanism in a simple manner. The nonlinear electrostatic gyrokinetic equation with proper conservation laws in general geometry is given by Eqs. (19), (21), and (22) of Ref. 19:

$$\frac{\partial F}{\partial t} + \frac{d\mathbf{R}}{dt} \cdot \boldsymbol{\nabla} F + \frac{dv_\parallel}{dt}\frac{\partial F}{\partial v_\parallel} = 0, \tag{3}$$

with

$$\frac{d\mathbf{R}}{dt} = v_\parallel \frac{\mathbf{B}^*}{B^*} + \frac{c\mathbf{b}}{e_i B^*} \times [e_i \boldsymbol{\nabla}\langle\langle\delta\phi\rangle\rangle + m_i\mu\boldsymbol{\nabla}B], \tag{4}$$

and

$$\frac{dv_\parallel}{dt} = -\frac{\mathbf{B}^*}{m_i B^*} \cdot [e_i \boldsymbol{\nabla}\langle\langle\delta\phi\rangle\rangle + m_i\mu\boldsymbol{\nabla}B]. \tag{5}$$

Here, the gyrokinetic Vlasov equation, Eq. (3) is written in terms of the guiding center distribution function $F(\mathbf{R},\mu,v_\parallel,t)$, with $\mu \equiv v_\perp^2/2B$. B^* is defined by

$$B^* \equiv \mathbf{b}\cdot\mathbf{B}^* = B + \frac{m_i c}{e_i}\mathbf{b}\cdot\boldsymbol{\nabla}\times v_\parallel \mathbf{b},$$

and is the phase-space volume in guiding center coordinates, i.e., the Jacobian of the transformation from the particle coordinates $(\mathbf{x},\mathbf{v})$ to the guiding center coordinates $(\mathbf{R},\mu,v_\parallel)$, satisfying Liouville's theorem

$$\boldsymbol{\nabla}\cdot\left(B^*\frac{d\mathbf{R}}{dt}\right) + \frac{\partial}{\partial v_\parallel}\left(B^*\frac{dv_\parallel}{dt}\right) = 0.$$

In the expression for $\mathbf{B}^* \equiv \mathbf{B} + (m_i c/e_i) v_\parallel \nabla\times\mathbf{b}$, the second term is typically ignored for stability and transport calculations. Since its magnitude is small, including this term will only make quantitative corrections to the linear growth rate and the turbulence-induced "diffusion" coefficients for tokamak plasmas which are mostly determined by other larger terms, such as the familiar ITG curvature drive. However, we find that keeping this correction is essential to identifying a new pinch mechanism in toroidal geometry. The question of the pinch's effect on coupling of drift/ITG modes to parallel shear flow drive will be left for future study.

The evolution equation for $\delta(nU_\parallel)$ can be obtained by taking an appropriate velocity moment of the perturbed distribution function

$$\delta(nU_\parallel) \equiv 2\pi \int d\mu dv_\parallel B^* \delta f v_\parallel,$$

and using the perturbed version of Eqs. (3)–(5),

$$\frac{\partial\delta f}{\partial t} + \frac{d\mathbf{R}}{dt}\cdot\boldsymbol{\nabla}\delta f + \frac{dv_\parallel}{dt}\frac{\partial\delta f}{\partial v_\parallel} = -\frac{d\mathbf{R}^{(1)}}{dt}\cdot\boldsymbol{\nabla}F_0 - \frac{dv_\parallel^{(1)}}{dt}\frac{\partial F_0}{\partial v_\parallel}. \tag{6}$$

Here,

$$\frac{d\mathbf{R}^{(1)}}{dt} = \frac{c\mathbf{b}}{B^*}\times\nabla\langle\langle\delta\phi\rangle\rangle,$$

and

$$\frac{dv_\parallel^{(1)}}{dt} = -\frac{e_i}{m_i}\frac{\mathbf{B}^*}{B^*}\cdot\nabla\langle\langle\delta\phi\rangle\rangle.$$

The last term on the RHS of Eq. (6) shows that the parallel acceleration of gyrocenters in a strongly magnetized plasma depends not only on the $k_\parallel$ of the fluctuations (along the equilibrium $\mathbf{B}$), but also on the perturbed $\mathbf{E}\times\mathbf{B}$ velocity which couples to the magnetic curvature $\mathbf{b}\times(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b}$, orthogonal to $\mathbf{b}$. This follows from the identity $\boldsymbol{\nabla}\times\mathbf{b} = \mathbf{b}(\mathbf{b}\cdot\boldsymbol{\nabla}\times\mathbf{b}) + \mathbf{b}\times(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b}$, and the inequality $k_\parallel \ll k_\perp$.

After straightforward algebra, including integrations by parts, we obtain

$$\begin{aligned}\frac{D}{Dt}\delta(nU_\parallel) = &-c\mathbf{b}\times\boldsymbol{\nabla}\delta\phi\cdot\boldsymbol{\nabla}\left(2\pi\int d\mu dv_\parallel F_0 v_\parallel\right) - 2\mathbf{b}\\ &\times(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b}\cdot\boldsymbol{\nabla}\delta\phi\left(2\pi\int d\mu dv_\parallel F_0 v_\parallel\right)\\ &-\frac{n_i e_i}{m_i}\mathbf{B}\cdot\boldsymbol{\nabla}\delta\phi.\end{aligned} \tag{7}$$

Here, we have used a long wavelength approximation $k_\perp\rho_i \ll 1$, and $(D/Dt)\delta(nU_\parallel)$ is short-hand for the moment of the LHS of Eq. (6) to be discussed later. On the RHS of Eq. (7), the first term can be written as

$$\begin{aligned}&c\mathbf{b}\times\nabla\delta\phi\cdot\nabla\left(2\pi\int d\mu dv_\parallel F_0 v_\parallel\right)\\ &\quad\simeq c\mathbf{b}\times\nabla\delta\phi\cdot\nabla\left(\frac{n_0 U_0}{B}\right),\end{aligned} \tag{8}$$

where we have used the fact that

$$n_0 U_0 \equiv 2\pi\int d\mu dv_\parallel B^* F_0 v_\parallel \simeq 2\pi B\int d\mu dv_\parallel F_0 v_\parallel.$$

Therefore, the fluctuation $\delta(nU_\parallel)$ is driven not only by the radial gradient of n_0U_0, which leads to a diffusive radial flux, but also by the gradient of B^{-1}, which leads to a nondiffusive radial flux of the parallel momentum. Note that the latter term $n_0U_0\mathbf{b}\times\nabla\delta\phi\cdot\nabla(1/B)$ is explicitly proportional to n_0U_0, and therefore can be identified as a "pinch."

The second term of the RHS of Eq. (7) is

$$\begin{aligned}&-2\mathbf{b}\times(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b}\cdot\boldsymbol{\nabla}\delta\phi\left(2\pi\int d\mu dv_\parallel F_0 v_\parallel\right)\\ &\quad\simeq -2\frac{n_0U_0}{B}\mathbf{b}\times(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b}\cdot\nabla\delta\phi.\end{aligned} \tag{9}$$

Since this pinch in Eqs. (8) and (9) is driven by the magnetic field inhomogeneity (which is not a thermodynamic force), it must be of the "turbulent equipartition pinch" (TEP) type, rather than a thermoelectric pinch. For this reason, we call this the "TEP" flux which will be discussed further in Sec. IV. This TEP contribution to the radial flux

$$\Pi_{\text{Mom}} \equiv \left\langle \sum_{\mathbf{k}} \delta(nU_\parallel)_{\mathbf{k}} \delta v_{r\mathbf{k}}^* \right\rangle$$

can be written as

$$\Pi_{\text{Mom}}^{\text{TEP}} = n_0 U_0 \left\langle \sum_{\mathbf{k}} \text{Re}(\tau_{c\mathbf{k}}) \left(2\omega_{d\parallel\mathbf{k}} \frac{e\delta\phi_{\mathbf{k}}}{T_\parallel} + \omega_{d\perp\mathbf{k}} \frac{e\delta\phi_{\mathbf{k}}}{T_\perp} \right) \delta v_{r\mathbf{k}}^* \right\rangle, \tag{10}$$

where $\tau_{c\mathbf{k}}$ is the inverse of the propagator. The real part of $\tau_{c\mathbf{k}}$ designates the correlation time of turbulence, while $\delta v_r \equiv (c/B)\hat{\mathbf{e}}_r \cdot \mathbf{b} \times \nabla \delta\phi$ is the fluctuating radial $\mathbf{E} \times \mathbf{B}$ velocity. Here, $\omega_{d\parallel\mathbf{k}} \equiv (cT_\parallel/eB)\mathbf{b} \times (\mathbf{b} \cdot \nabla)\mathbf{b} \cdot \mathbf{k}$ is the curvature drift of thermal ions, while $\omega_{d\perp\mathbf{k}} \equiv (cT_\perp/eB^2)\mathbf{b} \times \nabla B \cdot \mathbf{k}$ is the grad-B drift of thermal ions. Here, $\mathbf{k}$ is the wave vector of the fluctuation $\delta\phi$. In our sign convention, $\omega_{d\perp\mathbf{k}}$ and $\omega_{d\parallel\mathbf{k}}$ are negative at the low-B side midplane. From Eq. (10), noting that $\omega_{d\parallel}, \omega_{d\perp} \propto k_\theta \cos\theta + k_r \sin\theta$, for circular magnetic surface model geometry, we can see that the contribution of Eq. (10) to the radial flux of parallel momentum almost vanishes for flute-like fluctuations with nearly uniform intensity along $\mathbf{B}$.

There are other contributions to the "momentum pinch" which arise from the fact that both the curvature drift and the grad-B drift depend on $v_\parallel$ and μ of the ions, respectively. This can be traced back to the LHS of Eq. (6), where $(d\mathbf{R}/dt) \cdot \nabla \delta f$ contains an advection of δf by the velocity-dependent curvature drift $\mathbf{v}_{\text{curv}} \equiv (cm_i v_\parallel^2/e_i B)\mathbf{b} \times (\mathbf{b} \cdot \nabla)\mathbf{b}$ and the μ-dependent grad-B drift $\mathbf{v}_{\nabla B} \equiv cm_i\mu/e_i B\mathbf{b} \times \nabla B$ contained in the expression on the RHS of Eq. (4). Then, after taking the moment

$$\delta(nU_\parallel) \equiv 2\pi \int d\mu dv_\parallel B^* \delta f v_\parallel,$$

we obtain

$$\Pi_{\text{Mom}}^{\text{CTh}} = n_0 U_0 \left\langle \sum_{\mathbf{k}} \text{Re}\, \tau_{c\mathbf{k}} \left(3\omega_{d\parallel\mathbf{k}} \frac{\delta T_{\parallel\mathbf{k}}}{T_\parallel} + \omega_{d\perp\mathbf{k}} \frac{\delta T_{\perp\mathbf{k}}}{T_\perp} \right) v_{r\mathbf{k}}^* \right\rangle. \tag{11}$$

Here, "CTh" stands for the "curvature-driven thermoelectric" pinch. The reason for this acronym is that this portion of the off-diagonal flux is ultimately ∇T_i-driven. Then, one can write the final expression for the total radial flux of parallel momentum density as

$$\Pi_{\text{Mom}} = \Pi_{\text{Mom}}^{\text{Diff}} + \Pi_{\text{Mom}}^{\text{TEP}} + \Pi_{\text{Mom}}^{\text{CTh}} + \Pi_{\text{Mom}}^{\text{Acous}}, \tag{12}$$

where $\Pi_{\text{Mom}}^{\text{TEP}}$ and $\Pi_{\text{Mom}}^{\text{CTh}}$ are the new pinch contributions to the radial flux as given in Eqs. (10) and (11). The diffusive flux of the momentum density is given by

$$\Pi_{\text{Mom}}^{\text{Diff}} = -\left\langle \sum_{\mathbf{k}} \text{Re}\, \tau_{c\mathbf{k}} |\delta v_{r\mathbf{k}}|^2 \mathbf{e}_r \cdot \nabla(n_0 U_0) \right\rangle, \tag{13}$$

with a corresponding parallel momentum density diffusivity

$$\chi_{\text{Mom}} = \left\langle \sum_{\mathbf{k}} \text{Re}\, \tau_{c\mathbf{k}} |\delta v_{r\mathbf{k}}|^2 \right\rangle.$$

Note that this expression is similar to the test particle diffusion coefficient, and includes possible variations of τ_c which depend on the theoretical model. This is the main reason that the ratio between χ_ϕ and χ_i, known as the Prandtl number, varies depending on the theoretical model under study.[21,42–44] From experiments, while $\chi_\phi \sim \chi_i$ was typically observed,[1] some significant deviation between these two quantities began to emerge in recent years.[13,45,46]

Finally, $\Pi_{\text{Mom}}^{\text{Acous}}$ is a contribution from the third term on the RHS of Eq. (7). This is proportional to $k_\parallel$, related to the acoustic dynamics (from which we adopted a superscript), and leads to an off-diagonal nondiffusive flux if the $\mathbf{E} \times \mathbf{B}$ shear is included in the analysis as discussed in Ref. 15. $\Pi_{\text{Mom}}^{\text{Acous}}$ is produced when the $\mathbf{E} \times \mathbf{B}$ shear breaks the $x \to -x$ symmetry of the fluctuation spectrum about the resonant surface where $\mathbf{k} \cdot \mathbf{B} = 0$. The symmetry breaking mechanism considered in this paper and that considered in Ref. 15 which are necessary for net acceleration of plasmas along the magnetic field, can be considered as two components of a more general, unifying $\mathbf{B}^*$-symmetry breaking mechanism. Their relationships are summarized and unified in Table I.

Now, regarding the new turbulent convective ("TurCo") pinch terms, with the definition

$$\Pi_{\text{Mom}}^{\text{TurCo}} = \Pi_{\text{Mom}}^{\text{TEP}} + \Pi_{\text{Mom}}^{\text{CTh}} \equiv n_0 U_0 V_p^{\text{Mom}},$$

"the momentum pinch velocity," V_p^{Mom}, is given by

$$V_p^{\text{Mom}} = \left\langle \sum_{\mathbf{k}} \text{Re}\, \tau_{c\mathbf{k}} \left(2\omega_{d\parallel\mathbf{k}} \frac{e\delta\phi_{\mathbf{k}}}{T_\parallel} + \omega_{d\perp\mathbf{k}} \frac{e\delta\phi_{\mathbf{k}}}{T_\perp} + 3\omega_{d\parallel\mathbf{k}} \frac{\delta T_{\parallel\mathbf{k}}}{T_\parallel} + \omega_{d\perp\mathbf{k}} \frac{\delta T_{\perp\mathbf{k}}}{T_\perp} \right) v_{r\mathbf{k}}^* \right\rangle. \tag{14}$$

Note that, for a simple circular concentric high aspect ratio tokamak equilibrium, $\omega_{d\parallel}, \omega_{d\perp} \propto k_\theta \cos\theta + k_r \sin\theta = k_\theta \cos(\eta) + \hat{s}(\eta - \eta_0)\sin(\eta)$, in the ballooning coordinate η. With contributions from both normal curvature $[\propto \cos(\eta)]$ and geodesic curvature $[\propto (\eta - \eta_0)\sin(\eta)]$, ballooning fluctuations can produce a nonvanishing momentum pinch velocity even after flux-surface averaging. This will be illustrated at the end of Sec. III, with some examples of numerical evaluation of these quantities for profiles from experiments.

III. MOMENT ANALYSIS OF PARALLEL ANGULAR MOMENTUM TRANSPORT

In this section, we present a formal derivation of the turbulence driven radial flux of the parallel angular momentum density which we construct by taking moments of the nonlinear gyrokinetic equation. The final expression can be cast in a form in which not only the new momentum pinch terms are clearly identified, but also the physics mechanisms behind the curvature driven particle pinch are manifested transparently.

TABLE I. $\mathbf{B}^*$-symmetry breaking unifies two mechanisms: From $m_i B^* dv_\parallel/dt = -(e_i\mathbf{B} + m_i c v_\parallel \nabla\times\hat{b})\cdot\nabla\delta\phi$ (cf. Ref. 19).

	Gurcan *et al.*[a]	This paper
Net acceleration of parallel flow	$-e_i B\nabla_\parallel\delta\phi$	$-m_i c v_\parallel \nabla\times\hat{b}\cdot\nabla\delta\phi$
Symmetry-breaking	$k_\parallel$ over the spectral width	curvature drift $\sim\hat{b}\times(\hat{b}\cdot\nabla)\hat{b}$ over the flux surface
Provided by	mean $\mathbf{E}\times\mathbf{B}$ shear shifting fluctuations radially	ballooning mode structure causing finite net parallel acceleration over the flux surface
Main consequence	off-diagonal stress driven by $\mathbf{E}\times\mathbf{B}$ shear (or $\nabla P_i/n_i$ and velocity shear via radial force balance)	convective pinch-like term (The TEP-like piece is insensitive to mode details)
Most likely to be relevant for	plasmas with strong $\mathbf{E}\times\mathbf{B}$ shear, including H-mode, ITB	pinch is likely to be inward for OH and electron-heated plasmas

[a]Reference 15.

We can derive the nonlinear evolution of the parallel momentum density per ion mass:

$$nU_\parallel \equiv 2\pi\int d\mu dv_\parallel B^* F v_\parallel,$$

by taking a moment of the nonlinear gyrokinetic equation, Eq. (3). It is more convenient to use a conservative form of the nonlinear gyrokinetic equation [Eq. (24) of Ref. 19]

$$\frac{\partial(FB^*)}{\partial t} + \boldsymbol{\nabla}\cdot\left(FB^*\frac{d\mathbf{R}}{dt}\right) + \frac{\partial}{\partial v_\parallel}\left(FB^*\frac{dv_\parallel}{dt}\right) = 0. \tag{15}$$

Multiplying Eq. (15) by $v_\parallel$ and integrating over the velocity space, we obtain the following expression after some algebra,

$$\begin{aligned}\frac{1}{2\pi}\frac{\partial}{\partial t}nU_\parallel \equiv \frac{\partial}{\partial t}\int d\mu dv_\parallel B^* F v_\parallel = \\ -\int d\mu dv_\parallel\bigg(v_\parallel^2 B^*\mathbf{b}\cdot\boldsymbol{\nabla}F + \frac{m_i c}{e_i}\mathbf{b} \\ \times(\mathbf{b}\cdot\boldsymbol{\nabla})\mathbf{b}\cdot\boldsymbol{\nabla}F v_\parallel^3\bigg) - c\int d\mu dv_\parallel\boldsymbol{\nabla} \\ \times\mathbf{b}\cdot\left(\boldsymbol{\nabla}\phi + \frac{m_i\mu}{e_i}\boldsymbol{\nabla}B\right)Fv_\parallel - c\int d\mu dv_\parallel\mathbf{b} \\ \times\left(\boldsymbol{\nabla}\phi + \frac{m_i\mu}{e_i}\boldsymbol{\nabla}B\right)\cdot\boldsymbol{\nabla}Fv_\parallel \\ -\frac{e_i}{m_i}\int d\mu dv_\parallel F\mathbf{B}^*\cdot\boldsymbol{\nabla}\left(\phi + \frac{m_i\mu}{e_i}B\right).\end{aligned} \tag{16}$$

In this work, we consider a case in which the mean parallel velocity U_0 is lower than the phase velocity, $\omega/k_\parallel$, of the fluctuations such that its contribution to the propagator for the distribution function can be ignored. Quantitatively, this implies

$$\frac{k_\parallel U_0}{\omega} \sim \frac{k_\parallel U_0}{k_\theta\rho_s}\frac{a}{C_s} \sim M_s\frac{k_\parallel}{k_\theta}\frac{a}{\rho_s} \sim M_s\frac{a}{qR}\frac{1}{k_\theta\rho_s} < 1,$$

with the Mach number using the sound speed $M_s \equiv U_0/C_s$. Also, we adopt an ordering $k_\theta\rho_s > (a/qR)M_s$, and we assume $M_s < 1$ so that we can ignore $\mathbf{B}\cdot\boldsymbol{\nabla}nU_\parallel^2$ in comparison to $\mathbf{B}\cdot\boldsymbol{\nabla}P_\parallel$. The pressure moments per unit mass are defined as follows:

$$P_\parallel \equiv 2\pi\int d\mu dv_\parallel B^* F(v_\parallel - U_\parallel)^2, \tag{17}$$

$$P_\perp \equiv 2\pi\int d\mu dv_\parallel B^* F\mu B. \tag{18}$$

With this ordering, we can make the following simplifications. From the first term on the RHS of Eq. (16), we have

$$\begin{aligned}\int d\mu dv_\parallel F v_\parallel^3 = \int d\mu dv_\parallel F\{(v_\parallel - U_\parallel)^3 + 3(v_\parallel - U_\parallel)^2 U_\parallel \\ + 3(v_\parallel - U_\parallel)U_\parallel^2 + U_\parallel^3\} \simeq \frac{3}{2\pi}\frac{P_\parallel U_\parallel}{B}.\end{aligned} \tag{19}$$

Here, terms proportional to $U_\parallel^3$ and to a moment of $v_\parallel - U_\parallel$ have been ignored according to the ordering $M_s \ll 1$ and to the definition of $U_\parallel$, respectively. In addition, a term proportional to a moment of $(v_\parallel - U_\parallel)^3$ has been ignored by adopting a simple closure approximation. From similar considerations, the second term of the RHS of Eq. (16) can be approximated as follows, by using Eq. (18) and adopting a simple closure ignoring $\int d\mu dv_\parallel B^* F\mu(v_\parallel - U_\parallel)$:

$$\int d\mu dv_\parallel F\mu(v_\parallel - U_\parallel + U_\parallel) \simeq \frac{1}{2\pi}\frac{P_\perp U_\parallel}{B^2}. \tag{20}$$

Manipulations involving other terms in Eq. (16) are relatively straightforward, and employ the same vector identity and $\mathbf{k}$-component ordering utilized previously. Since $B^* \equiv \mathbf{b}\cdot\mathbf{B}^* = B + (m_i c/e_i)v_\parallel\mathbf{b}\cdot\nabla\times\mathbf{b}$, the scalar B^* can be approximated by B, ignoring a correction typically of the order of ρ_i/L_s, where $L_s = qR/\hat{s}$ is the shear length. While this term can be non-negligible very near the last closed flux surface of diverted plasmas where the magnetic shear $\hat{s}$ diverges more strongly than the magnetic safety factor q,[47] we ignore this term in this work.

We will also eventually ignore terms which are proportional to the gradient of B along $\mathbf{B}$, i.e., $\mathbf{B}\cdot\nabla B$ related to the mirror force. For instance, from the first term of the RHS of Eq. (16), we can show that, after an integration by parts,

$$-\int d\mu dv_\parallel (v_\parallel^2 B^* \mathbf{b}\cdot\nabla F) = -\frac{1}{2\pi}\mathbf{b}\cdot\nabla P_\parallel + \int d\mu dv_\parallel (v_\parallel^2 F \mathbf{b}\cdot\nabla B^* + \mathbf{b}\cdot\nabla n U_\parallel^2). \tag{21}$$

As mentioned before, we ignore the last term, $\mathbf{b}\cdot\nabla n U_\parallel^2$, assuming $M_s < 1$. The second term on the RHS of Eq. (21) is $\simeq (P_\parallel/B^2)\mathbf{B}\cdot\nabla B$. On the other hand, from the last term of Eq. (16),

$$-\frac{1}{m_i}\int d\mu dv_\parallel F \mathbf{B}\cdot\nabla m_i \mu B \simeq -\mathbf{B}\cdot\nabla B\int d\mu dv_\parallel F\mu \simeq -\frac{P_\perp}{2\pi B^2}\mathbf{B}\cdot\nabla B.$$

After being combined with $(P_\parallel/B^2)\mathbf{B}\cdot\nabla B$, this term leads to a familiar expression which is due to ion pressure anisotropy, which is in turn related to parallel viscosity,

$$\frac{\partial}{\partial t} n U_\parallel \equiv 2\pi \frac{\partial}{\partial t}\int d\mu dv_\parallel B^* F v_\parallel = \ldots - \frac{P_\perp - P_\parallel}{B^2}\mathbf{B}\cdot\nabla B. \tag{22}$$

While the term on the RHS can affect the long term evolution of the parallel momentum, in this paper, we focus on the turbulence driven radial transport of the parallel momentum. Therefore, we do not further discuss the effects of the contribution given in Eq. (22). Finally, we take the long wavelength limit ($k_\perp \rho_i \ll 1$) in this section, such that $\langle\langle\phi\rangle\rangle \simeq \phi$, in order to further elucidate the physics without the complications of keeping Bessel functions originating from the finite Larmor radius (FLR) effects.

With these considerations, we can write a nonlinear evolution equation for the parallel momentum, starting from Eq. (16), that is:

$$\frac{\partial}{\partial t}(m_i n U_\parallel) = -c\mathbf{b}\times\nabla\delta\phi\cdot\nabla\left(\frac{m_i n U_\parallel}{B}\right) - \frac{2m_i c n}{B}U_\parallel \mathbf{b} \times(\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\nabla\delta\phi - \frac{m_i^2 c}{e_i}\mathbf{b} \times\nabla B\cdot\nabla\left(\frac{P_\perp U_\parallel}{B^2}\right) - 3\frac{m_i^2 c}{e_i}\mathbf{b} \times(\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\nabla\left(\frac{P_\parallel U_\parallel}{B}\right) - n_i e_i \mathbf{b}\cdot\nabla\delta\phi - m_i \mathbf{b}\cdot\nabla P_\parallel. \tag{23}$$

In a low-β limit where the curvature drift and the ∇B drift are approximately equal, the form of Eq. (23) can be further simplified into a suggestive form illuminating the underlying physics. In low-β plasmas, $\mathbf{b}\times(\mathbf{b}\cdot\nabla)\mathbf{b} = (\nabla\times\mathbf{b})_\perp \simeq -\mathbf{B}\times\nabla(1/B)$, since $(\nabla\times\mathbf{B})/B = (4\pi/c)\mathbf{J}/B \ll \mathbf{B}\times\nabla(1/B)$. With this approximation, Eq. (23) can be further simplified to:

$$\frac{\partial}{\partial t}\left(\frac{m_i n U_\parallel}{B^3}\right) = -\frac{c\mathbf{b}\times\nabla\delta\phi}{B}\cdot\nabla\left(\frac{m_i n U_\parallel}{B^3}\right) - m_i^2 c\frac{\mathbf{b}\times\nabla B}{e_i B^3}\cdot\nabla\left(\frac{P_\perp U_\parallel}{B^2}\right) - 3m_i^2 c\frac{\mathbf{b}\times\nabla B}{e_i B^4}\cdot\nabla\left(\frac{P_\parallel U_\parallel}{B}\right) - \frac{1}{B^3}n_i e\mathbf{b}\cdot\nabla\delta\phi - \frac{m_i}{B^3}\mathbf{b}\cdot\nabla P_\parallel. \tag{24}$$

It is noteworthy that the fluctuations in $nU_\parallel$ cannot only be driven by the radial gradient of $nU_\parallel$, which eventually leads to a diffusive radial flux, but also by the gradient of B^{-3}. This leads to a nondiffusive radial flux of the parallel momentum. This latter term, which is

$$nU_\parallel B^2 \mathbf{b}\times\nabla\delta\phi\cdot\nabla(1/B^3)$$

will be identified as the "turbulent equipartition pinch" proportional to $nU_\parallel$, in Sec. IV.

While the $\mathbf{E}\times\mathbf{B}$ flow is compressible in inhomogeneous plasmas (i.e., $\nabla\cdot\mathbf{u}_E \equiv \nabla\cdot(\frac{c\mathbf{b}\times\nabla\phi}{B}) \neq 0$), we can make a low-$\beta$ approximation, i.e.,

$$\nabla\cdot(\mathbf{u}_E B^2) = c\nabla\cdot(\mathbf{B}\times\nabla\phi) = c\nabla\times\mathbf{B}\cdot\nabla\phi = 4\pi\mathbf{J}\cdot\nabla\phi \ll B^2\nabla\cdot\mathbf{u}_E,$$

to illuminate the physics associated with the compressibility caused by inhomogeneous B. After some manipulations using the low-β approximation, we can again rewrite Eq. (24) as follows:

$$\frac{\partial}{\partial t}\left(\frac{m_i n U_\parallel}{B}\right) = -\nabla\cdot\left(\frac{m_i n U_\parallel}{B}\mathbf{u}_E\right) - \frac{m_i^2 c}{e_i}\nabla\cdot\left[(\mathbf{b}\times\nabla B)\times\left(\frac{P_\perp U_\parallel}{B^3}\right)\right] - \frac{m_i^2 c}{e_i}\nabla\cdot\left[(3\mathbf{b}\times\nabla B)\times\left(\frac{P_\parallel U_\parallel}{B^3}\right)\right] - \frac{1}{B}n_i e_i\mathbf{b}\cdot\nabla\delta\phi - \frac{m_i}{B}\mathbf{b}\cdot\nabla P_\parallel. \tag{25}$$

It is important to recognize that the underlying symmetry and conservation laws of the nonlinear gyrokinetic equation in a nonuniform $\mathbf{B}$ field[19] lead to the particular combination of variables in Eq. (25) when one writes as many terms as possible in the form of a divergence of a flux.

First, since $B \propto 1/R$ in tokamaks, we note that

$$\frac{m_i n U_\parallel}{B} \propto m_i n U_\parallel R = m_i n R^2 \omega_\parallel \tag{26}$$

is the parallel angular momentum in tokamak geometry, with $\omega_\parallel$ being the parallel angular rotation frequency, and $m_i n R^2$

being the density of the moment of inertia. Therefore, within the context of this paper in which $U_\phi \simeq U_\parallel$, Eq. (25) describes the evolution of the toroidal angular momentum, $m_i n R^2 \omega_\phi$. The expression $(m_i n U_\parallel / B)\mathbf{u}_E \propto m_i n U_\parallel R \mathbf{u}_E \propto m_i n R^2 \omega_\phi \mathbf{u}_E$ is essentially a radial flux of the toroidal angular momentum. It's also noteworthy that this particular combination arose without assuming axisymmetry. Therefore, this formulation should be useful for future applications to three-dimensional systems, including quasiaxisymmetric stellarators such as the National Compact Stellarator Experiment (NCSX). It will be interesting to contrast this to a neoclassical approach considering the electrostatic fluctuation ripples.[48]

Typically, transport analyses[49] deal with the temporal evolution of the flux-surface-averaged toroidal angular momentum density $\langle m_i n R^2 \rangle \omega_\phi$, where the toroidal angular frequency is a flux function. In this paper, we use a set of variables (ψ, θ, ζ) to denote the radial, poloidal, and toroidal coordinates, respectively. The equilibrium magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = \nabla\zeta \times \nabla\psi + I(\psi)\nabla\zeta, \tag{27}$$

where $d\psi = RB_\theta dr$, and the toroidal magnetic field strength is given by $B_\phi = I(\psi)/R$. From Eq. (27), we can also show that the following useful identity holds,

$$R^2 \nabla\zeta = \nabla\psi \times \mathbf{B}/B^2 + I(\psi)\mathbf{B}/B^2. \tag{28}$$

With these definitions, the mean toroidal angular momentum density evolution equation can be derived by taking a flux-surface-average of Eq. (25), after multiplying by $B_0 R_0$ to restore the proper dimensions, assuming $\omega_\parallel = \omega_\parallel(\psi)$, i.e.,

$$\frac{\partial}{\partial t}(\langle m_i n R^2 \rangle \omega_\parallel) = -\langle \nabla \cdot \mathbf{\Pi}_{Ang} \rangle - \langle \nabla \cdot \mathbf{\Pi}_{Geo} \rangle + \langle \mathbf{B} \cdot \mathbf{T}_\parallel \rangle. \tag{29}$$

Here, the first term on the RHS of Eq. (29),

$$\mathbf{\Pi}_{Ang} \equiv m_i \delta(n U_\parallel R) \frac{c\mathbf{b} \times \nabla\delta\phi}{B},$$

with $\delta(nU_\parallel R) \equiv 2\pi \int d\mu dv_\parallel B^* \delta f R v_\parallel$, is the main turbulence driven contributor to the evolution of the mean angular momentum, i.e., the perturbed parallel angular momentum density carried by the fluctuating $\mathbf{E} \times \mathbf{B}$ velocity due to turbulence. Note that this expression contains a nonradial, perpendicular component, as well as the radial component of the fluctuating $\mathbf{E} \times \mathbf{B}$ velocity. However, using Gauss theorem, one can show that,[50] for any vector field $\mathbf{A}$,

$$\langle \nabla \cdot \mathbf{A} \rangle = \frac{1}{V'} \frac{\partial}{\partial\psi} V' \langle \mathbf{A} \cdot \nabla\psi \rangle,$$

where V is the volume element of the flux-tube, $V' \equiv dV/d\psi$. Therefore, only the radial component of $\mathbf{A}$ contributes to the flux-surface-average of the divergence of $\mathbf{A}$. Thus, we obtain,

$$\begin{aligned}\langle \nabla \cdot \mathbf{\Pi}_{\rm Ang}^{\rm Turb} \rangle &= \frac{1}{V'} \frac{\partial}{\partial\psi} [V' \langle \mathbf{\Pi}_{\rm Ang} \cdot \nabla\psi \rangle] \\ &= \frac{1}{V'} \frac{\partial}{\partial\psi} \left[V' \left\langle m_i \delta(nU_\parallel R) \frac{c}{B} \mathbf{b} \times \nabla\delta\phi^* \cdot \nabla\psi \right\rangle \right] \\ &\simeq \frac{1}{V'} \frac{\partial}{\partial\psi} \left[V' \left\langle m_i c R \sum_{\mathbf{k}} \delta(nU_\parallel R)_{\mathbf{k}} \frac{\partial}{\partial\zeta} \delta\phi_{\mathbf{k}}^* \right\rangle \right].\end{aligned} \tag{30}$$

Here, we used the fact that $k_\parallel \ll k_\perp$, and the identity given in Eq. (28).

The second term on the RHS of Eq. (29) has not been considered in previous studies of anomalous momentum transport. Its turbulent contribution,

$$\mathbf{\Pi}_{\rm Geo}^{\rm Turb} \equiv \delta(nU_\parallel R)(\mathbf{b} \times \nabla B) \left(\frac{\delta T_\perp + 3\delta T_\parallel}{B^2} \right) \tag{31}$$

can be considered as the parallel angular momentum density advected by the velocity-dependent residual part of the curvature drift (which has been replaced by the grad-B drift within the low-β approximation). We denote this as $\mathbf{\Pi}_{\rm Geo}^{\rm Turb}$, since the flux-surface-averaged value of its divergence is proportional to the geodesic curvature in the low-β approximation, i.e., the flux surface component of the magnetic field line curvature,[51]

$$(\mathbf{b} \cdot \nabla)\mathbf{b} \times \mathbf{B} \cdot \nabla\psi \simeq \frac{1}{B} \nabla B \times \mathbf{B} \cdot \nabla\psi.$$

Since the flux-surface-average of its divergence contains the expression $\langle (\cdots) \mathbf{B} \times \nabla B \cdot \nabla\psi / B^2 \rangle$, then by using Eq. (28) and axisymmetric equilibrium, one can show that this is proportional to $(I(\psi)/B^2)(\mathbf{B} \cdot \nabla)B$. Therefore, this contribution is subdominant to the first term on the RHS of Eq. (29), which is the main term we keep in this paper. A more formal estimation using a quasilinear expansion in terms of δT and $\delta(nU_\parallel R)$ also shows that the $\langle \mathbf{\Pi}_{\rm Geo}^{\rm Turb} \cdot \nabla\psi \rangle$ term is $o(\omega_{di}/\omega)$ smaller than the turbulent convective pinch terms which originate from $\langle \mathbf{\Pi}_{\rm Ang}^{\rm TurCo} \cdot \nabla\psi \rangle$. However, the mathematical form of these terms as functions of thermodynamic driving forces is different from those of either diffusive or turbulent convective pinch terms. This subdominant term should not be confused with the curvature driven thermoelectric flux $\mathbf{\Pi}^{\rm CTh}$ in Eq. (11) and in Eq. (66), which originates from $\mathbf{\Pi}_{\rm Ang}$.

Finally, noting that for any scalar S, $\langle \mathbf{B} \cdot \nabla S \rangle = 0$, we observe that the surviving contributions from the last term

$$\mathbf{B} \cdot \mathbf{T}_\parallel \equiv -\mathbf{B} \cdot \frac{R}{B}(n_i e_i \nabla\phi + \nabla P_\parallel),$$

involving the parallel torque $\mathbf{T}_\parallel$ in Eq. (29), are proportional to $(\mathbf{B} \cdot \nabla)B$, or $k_\parallel$ of the fluctuations.[52] As mentioned before, the effects associated with these parallel dynamics are not addressed in this paper. The physics associated with the symmetry breaking of $k_{\|}$ has been extensively discussed in Ref. 15.

For the evaluation of the nonlinear turbulent flux of angular momentum $\mathbf{\Pi}_{Ang}^{Turb}$ in Eq. (30), the expression for the perturbed angular momentum $\delta(nU_{\|}R)$ can be obtained by linearizing Eq. (23). In $\mathbf{k}$-space, it can be written as

$$\begin{aligned}[-i\omega_{\mathbf{k}} + \Delta\omega_{\mathbf{k}} + i(3\omega_{d\|\mathbf{k}} + \omega_{d\perp\mathbf{k}})]\delta(nU_{\|}R)_{\mathbf{k}} \\ = -\delta v_{r\mathbf{k}}\hat{\mathbf{e}}_{\psi}\cdot\nabla(n_0U_0R) - i2\omega_{d\|\mathbf{k}}\frac{e\delta\phi_{\mathbf{k}}}{T_{\|}}n_0U_0R \\ - i\left(3\omega_{d\|\mathbf{k}}\frac{\delta T_{\|\mathbf{k}}}{T_{\|}} + \omega_{d\perp\mathbf{k}}\frac{\delta T_{\perp\mathbf{k}}}{T_{\perp}}\right)n_0U_0R \\ - ik_{\|}R(\delta P_{\|\mathbf{k}} + n_0e_i\delta\phi_{\mathbf{k}}). \end{aligned} \tag{32}$$

The origin of various terms has been discussed in Sec. II, in relation to Eqs. (10) and (11). The expression multiplying $\delta(nU_{\|}R)_{\mathbf{k}}$ on the LHS of Eq. (23) is the $(\mathbf{k},\omega)$-space version of the renormalized propagator, in which $\Delta\omega_{\mathbf{k}}$ is the decorrelation rate which originates from the $\mathbf{E}\times\mathbf{B}$ nonlinear term in Eq. (32). Here, we consider stationary turbulence ($\gamma_{\mathbf{k}}=0$), but with a finite amplitude and thus, a finite correlation time. $\Delta\omega_{\mathbf{k}}$ is from the $\mathbf{E}\times\mathbf{B}$ nonlinearity-induced self-decorrelation rate. Note that causality requires that $\Delta\omega_{\mathbf{k}}>0$. For rough estimates, it is useful to take $\Delta\omega_{\mathbf{k}}\sim|\gamma_{lin,\mathbf{k}}|$. The absolute value applies for the case of damped modes (i.e., nonresonant quasilinear diffusion is positive definite). Here, $\tau_{c\mathbf{k}}\equiv[-i\omega_{\mathbf{k}}+\Delta\omega_{\mathbf{k}}+i(3\omega_{d\|\mathbf{k}}+\omega_{d\perp\mathbf{k}})]^{-1}$ is the inverse of the propagator. Its real part, which is positive definite and independent of mode propagation direction, corresponds to the correlation time of the turbulence.

Now, we can explicitly evaluate the angular momentum flux and can calculate its divergence from Eq. (30). From the first term on the RHS of Eq. (32), we obtain the usual diffusive part of the radial component of the toroidal angular momentum density flux:

$$\begin{aligned}\langle\mathbf{\Pi}_{\rm Ang}^{\rm Diff}\cdot\nabla\psi\rangle &= -\left\langle\sum_{\mathbf{k}}\mathrm{Re}\,\tau_{c\mathbf{k}}|\delta v_{r\mathbf{k}}|^2\nabla(m_in_0U_0R)\cdot\nabla\psi\right\rangle \\ &= -\chi_{\rm Ang}\left\langle(RB_{\theta})^2\frac{\partial}{\partial\psi}(m_in_0R^2\omega_{\|})\right\rangle. \end{aligned} \tag{33}$$

While one can measure the angular momentum density flux directly from nonlinear turbulence simulations, transport analysis[49] of experimental data involves flux-surface-averaged quantities. Here, the flux-surface-averaged "angular momentum density diffusivity" can be defined as

$$\begin{aligned}\chi_{\rm Ang} &\equiv \left\langle\sum_{\mathbf{k}}\mathrm{Re}\,\tau_{c\mathbf{k}}|\delta v_{r\mathbf{k}}|^2\right\rangle \\ &= \left\langle\left(\frac{c}{RB_{\theta}}\right)^2\sum_{\mathbf{k}}\mathrm{Re}\,\tau_{c\mathbf{k}}\ell^2|\delta\phi_{\mathbf{k}}|^2\right\rangle. \end{aligned} \tag{34}$$

To obtain Eq. (34), we used the following identities: $|\nabla\psi|=RB_{\theta}$, $\mathbf{b}\times\hat{\mathbf{e}}_{\psi}\cdot\mathbf{k}=\ell B/RB_{\theta}$, $\delta v_{r\mathbf{k}}=-i(c\ell/RB_{\theta})\delta\phi_{\mathbf{k}}$ with ℓ=toroidal mode number. From the second term on the RHS of Eq. (32), we obtain the TEP part of the radial component of the toroidal angular momentum density flux, i.e.,

$$\begin{aligned}&\langle\mathbf{\Pi}_{\rm Ang}^{\rm TEP}\cdot\nabla\psi\rangle \\ &= -2\left\langle\sum_{\mathbf{k}}\mathrm{Re}\,\tau_{c\mathbf{k}}\delta v_{r\mathbf{k}}^{*}i\left(\omega_{d\|\mathbf{k}}\frac{e\delta\phi_{\mathbf{k}}}{T_{\|}}\right)m_in_0R^2\omega_{\|}RB_{\theta}\right\rangle \\ &= \langle m_in_0R^3B_{\theta}\rangle\omega_{\|}V_{\rm Ang}^{\rm TEP}. \end{aligned} \tag{35}$$

Here, the flux-surface-averaged "TEP angular momentum pinch" can be defined as

$$\begin{aligned}V_{\rm Ang}^{\rm TEP} &\equiv -2\left\langle\sum_{\mathbf{k}}i\,\mathrm{Re}\,\tau_{c\mathbf{k}}\delta v_{r\mathbf{k}}^{*}\omega_{d\|\mathbf{k}}\frac{e\delta\phi_{\mathbf{k}}}{T_{\|}}\right\rangle \\ &= 2\left\langle\frac{c}{RB_{\theta}}\sum_{\mathbf{k}}\mathrm{Re}\,\tau_{c\mathbf{k}}\,\ell\,\omega_{d\|\mathbf{k}}\frac{e}{T_{\|}}|\delta\phi_{\mathbf{k}}|^2\right\rangle. \end{aligned} \tag{36}$$

Using the identity $\omega_{d\|,\mathbf{k}}(0)=-(cT_{\|}/e_iRB_{\theta})\ell/R$ at the low-B side midplane ($\theta=0$), we can write

$$\begin{aligned}V_{\rm Ang}^{\rm TEP} &= -2\left\langle\frac{1}{R}\left(\frac{c}{RB_{\theta}}\right)^2\sum_{\mathbf{k}}\mathrm{Re}\,\tau_{c\mathbf{k}}\ell^2\frac{\omega_{d\|\mathbf{k}}(\theta)}{\omega_{d\|\mathbf{k}}(0)}|\delta\phi_{\mathbf{k}}|^2\right\rangle \\ &= -2\left\langle\frac{1}{R}\sum_{\mathbf{k}}\mathrm{Re}\,\tau_{c\mathbf{k}}\frac{\omega_{d\|\mathbf{k}}(\theta)}{\omega_{d\|\mathbf{k}}(0)}|\delta v_{r,\mathbf{k}}|^2\right\rangle. \end{aligned} \tag{37}$$

Note that, in comparison to Eq. (10) which gives the TEP pinch of the (linear) momentum density, the piece proportional to $\omega_{d\perp\mathbf{k}}$ is absent in Eq. (36). This is a consequence of the fact that the definition of angular momentum density has an additional factor of R in comparison to the definition of linear momentum density. Since $R\propto 1/B$, a part of the TEP pinch driven by ∇B for the momentum, as described by Eq. (8), does not exist for the angular momentum.

From the third term on the RHS of Eq. (32), we obtain the curvature driven thermoelectric pinch (CTh) part of the radial flux of the toroidal angular momentum,

$$\begin{aligned}\langle\mathbf{\Pi}_{\rm Ang}^{\rm CTh}\cdot\nabla\psi\rangle &= -\left\langle\sum_{\mathbf{k}}\mathrm{Re}\left[i\tau_{c\mathbf{k}}\delta v_{r\mathbf{k}}^{*}\left\{3\omega_{d\|\mathbf{k}}\frac{\delta T_{\|\mathbf{k}}}{T_{\|}}\right.\right.\right. \\ &\qquad\left.\left.\left. +\,\omega_{d\perp\mathbf{k}}\frac{\delta T_{\perp\mathbf{k}}}{T_{\perp}}\right\}\right]m_in_0R^2\omega_{\|}RB_{\theta}\right\rangle \\ &= \langle m_in_0R^3B_{\theta}\rangle\omega_{\|}V_{\rm Ang}^{\rm CTh}. \end{aligned} \tag{38}$$

Here, the flux-surface-averaged "CTh angular momentum density pinch" can be defined as

$$\begin{aligned}V_{\rm Ang}^{\rm CTh} &\equiv -\left\langle\sum_{\mathbf{k}}\mathrm{Re}\left[i\tau_{c\mathbf{k}}\delta v_{r\mathbf{k}}^{*}\left\{3\omega_{d\|\mathbf{k}}\frac{\delta T_{\|\mathbf{k}}}{T_{\|}} + \omega_{d\perp\mathbf{k}}\frac{\delta T_{\perp\mathbf{k}}}{T_{\perp}}\right\}\right]\right\rangle \\ &= \left\langle\frac{c}{RB_{\theta}}\sum_{\mathbf{k}}(\mathrm{Re}\,\tau_{c\mathbf{k}})\,\ell\left(3\omega_{d\|\mathbf{k}}\frac{\delta T_{\|\mathbf{k}}}{T_{\|}} + \omega_{d\perp\mathbf{k}}\frac{\delta T_{\perp\mathbf{k}}}{T_{\perp}}\right)\delta\phi_{\mathbf{k}}^{*}\right\rangle. \end{aligned} \tag{39}$$

Again, using the identity $\omega_{d\perp,\mathbf{k}}(0)=-(cT_{\perp}/e_iRB_{\theta})\ell/R$ at $\theta=0$, we can write

TABLE II. Here, $F_{\rm norma} \equiv \int d\theta \omega_{d-{\rm norma}}(\theta)|\delta\phi(\theta)|^2 h(\theta)/\int d\theta\omega_{d-{\rm norma}}(0)|\delta\phi(\theta)|^2 h(\theta)$ and $F_{\rm geo} \equiv \int d\theta\omega_{d-{\rm geo}}(\theta)|\delta\phi(\theta)|^2 h(\theta)/\int d\theta\omega_{d-{\rm norma}}(0)|\delta\phi(\theta)|^2 h(\theta)$, where $\omega_{d-{\rm norma}}(\theta)$ and $\omega_{d-{\rm geo}}(\theta)$ are the normal and geodesic components, respectively, of the magnetic drift frequency calculated numerically using results of a MHD equilibrium code for noncircular cross section geometry, and $R(\theta)=R_0 h(\theta)$, where R is the major radius and R_0 is its average value for the chosen magnetic surface, and $\rho_i \equiv \sqrt{T_i/m_i}/(eB_0/m_i c)$. Note that $\langle\cdots\rangle \propto \int d\theta \mathcal{J} B \cdots \propto \int d\theta R^2(B_0/R)\cdots \propto \int d\theta h(\theta)\cdots$ for the Jacobian $\mathcal{J}\propto R^2$ chosen here. Also, $A=R/r$ is the aspect ratio, κ is the ellipticity, and δ is the triangularity.

Radial location	$r/a=0.4$	$r/a=0.7$
Key local values	$\hat{s}=0.78$, $T_i/T_e=1.17$	$\hat{s}=0.80$, $T_i/T_e=1.38$
from profiles and	$\eta_i^e=2.34$, $R_0/L_{ne}=9.13$	$\eta_i^e=1.72$, $R_0/L_{ne}=6.14$
MHD equilibrium	$A=11.7$, $\kappa=1.31$, $\delta=0.0335$	$A=6.81$, $\kappa=1.39$, $\delta=0.0734$
Complex frequency normalized to	$\gamma=1.71$	$\gamma=0.72$
(c_s/R_0) at $k_\theta\rho_i=0.50$	$\omega_r=-0.08$	$\omega_r=-0.50$
$F_{\rm norma}$	0.56	0.54
$F_{\rm geo}$	0.38	0.41

$$V_{\rm Ang}^{\rm CTh} = -3\left\langle \frac{1}{R}\left(\frac{c}{RB_\theta}\right)^2 \sum_{\mathbf{k}} {\rm Re}\ \tau_{c\mathbf{k}} \ell^2 \frac{\omega_{d\|\mathbf{k}}(\theta)}{\omega_{d\|\mathbf{k}}(0)} \frac{\delta T_\|}{e_i}\delta\phi_{\mathbf{k}}^* \right\rangle - \left\langle \frac{1}{R}\left(\frac{c}{RB_\theta}\right)^2 \sum_{\mathbf{k}} {\rm Re}\ \tau_{c\mathbf{k}} \ell^2 \frac{\omega_{d\perp\mathbf{k}}(\theta)}{\omega_{d\perp\mathbf{k}}(0)} \frac{\delta T_\perp}{e_i}\delta\phi_{\mathbf{k}}^* \right\rangle$$
$$= -3\left\langle \frac{1}{R}\sum_{\mathbf{k}} {\rm Re}\ \tau_{c\mathbf{k}} \frac{\omega_{d\|\mathbf{k}}(\theta)}{\omega_{d\|\mathbf{k}}(0)} \frac{\delta T_\|/e_i}{\delta\phi_{\mathbf{k}}}|\delta v_{r,\mathbf{k}}|^2 \right\rangle - \left\langle \frac{1}{R}\sum_{\mathbf{k}} {\rm Re}\ \tau_{c\mathbf{k}} \frac{\omega_{d\perp\mathbf{k}}(\theta)}{\omega_{d\perp\mathbf{k}}(0)} \frac{\delta T_\perp/e_i}{\delta\phi_{\mathbf{k}}}|\delta v_{r,\mathbf{k}}|^2 \right\rangle. \quad (40)$$

The last term in Eq. (32) contributes nothing in the absence of mean $\mathbf{E}\times\mathbf{B}$ shear. In summary, the flux-surface-averaged turbulence-driven parallel angular momentum flux, in the absence of $\mathbf{E}\times\mathbf{B}$ shear, can be characterized as the sum of a "diffusive" flux and the "turbulent convective" flux,

$$\langle \mathbf{\Pi}_{\rm Ang}^{\rm Turb}\cdot\nabla\psi\rangle = \langle \mathbf{\Pi}_{\rm Ang}^{\rm Diff}\cdot\nabla\psi\rangle + \langle \mathbf{\Pi}_{\rm Ang}^{\rm TurCo}\cdot\nabla\psi\rangle = -\chi_{\rm Ang}\left\langle (RB_\theta)^2 \frac{\partial}{\partial\psi}(m_i n_0 R^2 \omega_\|)\right\rangle + V_{\rm Ang}^{\rm TurCo}\langle m_i n_0 R^3 B_\theta\rangle \omega_\|. \quad (41)$$

Here, the angular momentum diffusivity $\chi_{\rm Ang}$ is given by Eq. (34), and the turbulent convective (TurCo) pinch velocity is given by

$$V_{\rm Ang}^{\rm TurCo} = V_{\rm Ang}^{\rm TEP} + V_{\rm Ang}^{\rm CTh},$$

with the TEP contribution and the CTh contribution given by Eq. (37) and Eq. (40), respectively. From Eqs. (34), (37), and (40), it is obvious that the relative magnitude of the pinch velocity $V_{\rm Ang}^{\rm TurCo}$ and the angular momentum density diffusivity $\chi_{\rm Ang}$ can be quantified in terms of two dimensionless parameters,

$$F_{\rm balloon} \equiv \frac{\langle \omega_{d\mathbf{k}}(\theta)|\delta\phi_{\mathbf{k}}(\theta)|^2\rangle}{\langle \omega_{d\mathbf{k}}(0)|\delta\phi_{\mathbf{k}}(\theta)|^2\rangle},$$

and

$$G^{\rm Th} \equiv \frac{\langle (\delta T_{i\mathbf{k}}/e_i)\delta\phi_{\mathbf{k}}^*\rangle}{\langle |\delta\phi_{\mathbf{k}}(\theta)|^2\rangle},$$

for quantities with subscripts, $_\|$ and $_\perp$. $F_{\rm balloon}$ quantifies the ballooning mode structure. We can distinguish the contributions from the normal curvature and the geodesic curvature, by defining

$$F_{\rm norma} \equiv \frac{\langle \omega_{d-{\rm norma}}(\theta)|\delta\phi(\theta)|^2\rangle}{\langle \omega_d(0)|\delta\phi(\theta)|^2\rangle},$$

and

$$F_{\rm geo} \equiv \frac{\langle \omega_{d-{\rm geo}}(\theta)|\delta\phi(\theta)|^2\rangle}{\langle \omega_d(0)|\delta\phi(\theta)|^2\rangle}.$$

We note that for outward ballooning mode structure, $F_{\rm norma} > 0$, and $F_{\rm geo} > 0$, for positive magnetic shear.

From the FULL code[53] calculation using positive magnetic shear parameters and profiles from JT-60U,[54] we find that the fluctuation is strongly ballooning outward, yielding $F_{\rm norma} \simeq 0.5$, and $F_{\rm geo} \simeq 0.4$ at two different radii, while the normalized growth rate varies more than a factor of 2. More thorough parameter scans will be reported in future publications (see Table II).

While evaluating $F_{\rm balloon}$ using the linear eigenmode structure as done here fits with the quasilinear approach in this paper, this might lead to an overestimate compared to that for nonlinearly saturated turbulence. It is commonly observed from long wavelength drift wave turbulence simulations[55] that strongly ballooning, radially elongated linear eigenmode structures are destroyed via random shearing[56,57] due to turbulence driven zonal flows. While in-out asymmetry of fluctuation amplitudes persists in the nonlinear regime, it might be weaker than that in the linear regime.

$G^{\rm Th}$ quantifies the relative strength of contributions from ion temperature fluctuations related to the curvature driven thermoelectric effect. Due to the phase relation between δT_i and $\delta\phi$, the sign of $G^{\rm Th}$ depends on the mode propagation direction. While $G^{\rm Th} > 0$ for fluctuations propagating in the

TABLE III. Summary of key expressions for pinch velocity of angular momentum density.

	Turbulent Convective (TurCo) Pinch Velocity	
Theoretical Model	V^{TEP} (turbulent equipartition)	V^{CTh} (curvature-driven thermoelectric)
Hydrodynamic	$-2\left\langle \frac{1}{R}\Sigma_{\mathbf{k}}\mathrm{Re}\ \tau_{c\mathbf{k}}\frac{\omega_{d\parallel\mathbf{k}}(\theta)}{\omega_{d\parallel\mathbf{k}}(0)}\left\vert\delta v_{r,\mathbf{k}}\right\vert^2\right\rangle$	$\left\langle \frac{c}{RB_\theta}\Sigma_{\mathbf{k}}(\mathrm{Re}\ \tau_{c\mathbf{k}})l\left[3\omega_{d\parallel\mathbf{k}}\frac{\delta T_{\parallel\mathbf{k}}}{T_\parallel}+\omega_{d\perp\mathbf{k}}\frac{\delta T_{\perp\mathbf{k}}}{T_\perp}\right]\delta\phi_{\mathbf{k}}^*\right\rangle$
Eq. (41)	Eq. (37)	Eq. (40)
Finite Larmor Radius Generalization	See Eq. (65)	See Eq. (67)
Typical magnitude	$-2\chi_{Ang}/R_0$	$-4G^{\mathrm{Th}}\chi_{\mathrm{Ang}}/R_0$
Comments	Always inward for outward ballooning turbulence for normal magnetic shear; Insensitive to details of microinstabilities	Sign of G^{Th} depends on mode propagation; Inward for TEM; Can be either inward or outward for ITG; Small for electron-heated or OH plasmas
Independent of	the sign of either B_ϕ or I_p	the sign of either B_ϕ or I_p

electron diamagnetic direction, an accurate prediction for fluctuations propagating in the ion diamagnetic direction is difficult due to a hydrodynamic approximation employed in the derivation. Using these two dimensionless quantities, we can write the pinch velocity in terms of the angular momentum density diffusivity,

$$V_{\mathrm{Ang}}^{\mathrm{TEP}} \simeq -\frac{2F_{\mathrm{balloon}}}{R_0}\chi_{\mathrm{Ang}}, \tag{42}$$

and

$$V_{\mathrm{Ang}}^{\mathrm{CTh}} \simeq -\frac{F_{\mathrm{balloon}}(3G_\parallel^{\mathrm{Th}}+G_\perp^{\mathrm{Th}})}{R_0}\chi_{\mathrm{Ang}}. \tag{43}$$

IV. PHYSICS OF THE CURVATURE DRIVEN PARALLEL ANGULAR MOMENTUM PINCH

In this section, we discuss the physics of the curvature driven pinch of parallel angular momentum density which was derived in Sec. III. Since the aim of this section is physical insight and understanding, rather than the presentation of detailed results, we use a simplified notation here. The reader seeking detailed results is referred to Secs. III and V, and to Table III.

As discussed previously, a unique feature of the turbulence driven convective pinch derived here is that it consists of pieces driven by both nonthermodynamic (i.e., ∇B) and thermodynamic (i.e., ∇T_i) forces. The nonthermodynamic force driven terms suggest a physical interpretation in terms of the theory of "turbulent equipartition" (TEP). In particular, we compare and contrast the pinch of parallel angular momentum with the now familiar TEP mechanism for the particle pinch. For the TEP particle pinch, the underlying conservation laws of the nonlinear gyrokinetic equation are the ultimate motivation for the TEP interpretation. A simple introduction to TEP fluxes and their relation to homogenization is presented in Appendix B, with an illustration of the TEP pinch for density in a 2D system[58] with a straight, but inhomogeneous magnetic field $\mathbf{B}=B(x,y)\hat{\mathbf{z}}$.

Our starting point is Eq. (24), which states that a "magnetically weighted" parallel momentum density $m_i n U_\parallel/B^3$ evolves according to

$$\frac{\partial}{\partial t}\left(\frac{m_i n U_\parallel}{B^3}\right)+\frac{c\mathbf{b}\times\nabla\delta\phi}{B}\cdot\boldsymbol{\nabla}\left(\frac{m_i n U_\parallel}{B^3}\right) = -m_i^2 c\frac{\mathbf{b}\times\nabla B}{B^3}\cdot\boldsymbol{\nabla}\left(\frac{P_\perp U_\parallel}{B^2}\right)-3m_i^2 c\frac{\mathbf{b}\times\nabla B}{B^4} \cdot\boldsymbol{\nabla}\frac{P_\parallel U_\parallel}{B}-\frac{1}{B^3}ne_i\mathbf{b}\cdot\nabla\delta\phi-\frac{m_i}{B^3}\mathbf{b}\cdot\boldsymbol{\nabla}P_\parallel. \tag{44}$$

Note that the first two terms on the RHS of Eq. (44) (i.e., related to ion curvature drift and ion pressure) are formally $O(\omega_{di}/\omega)=O(a/R)$ with respect to the LHS. Similarly, the second two terms on the RHS of Eq. (44) (i.e., related to parallel acoustic dynamics) are formally $O(k_\parallel/k_\perp)$ with respect to the LHS. Thus, to the lowest order in a/R and $k_\parallel/k_\perp$, the magnetically weighted parallel ion momentum density obeys the equation,

$$\frac{\partial}{\partial t}\left(\frac{m_i n U_\parallel}{B^3}\right)+\frac{c\mathbf{b}\times\boldsymbol{\nabla}\delta\phi}{B}\cdot\boldsymbol{\nabla}\left(\frac{m_i n U_\parallel}{B^3}\right)=0. \tag{45}$$

Note that the magnetically weighted angular momentum density is a locally advected scalar, so that the addition of any minute diffusive dissipation to the RHS of Eq. (45) will regularize it so that it becomes isomorphic to Eq. (B1), thus indicating that the scalar $m_i n U_\parallel/B^3$ will be turbulently mixed or "homogenized," given sufficient time. As discussed in Appendix B, such homogenization problems are prime candidates for the application of TEP theory. Before launching into a discussion of TEP theory for $m_i n U_\parallel/B^3$, we first comment that the approximate conservation of $m_i n U_\parallel/B^3 \propto m_i n U_\parallel R/B^2$ (since $B\propto 1/R$ in a torus) is a consequence of: (i) the fact that $B^2\mathbf{u}_E$ is an approximately incompressible flow velocity in the low-β toroidal equilibrium, and (ii) the fact that $m_i n U_\parallel R$, the parallel angular momentum density, is the "natural" quantity which is homogenized or mixed by the flow $B^2\mathbf{u}_E$. (i) and (ii) together explain the origin of the magnetically weighted momentum density $m_i n U_\parallel/B^3$ as the advected scalar to be homogenized. Note also that since $m_i n U_\parallel / B^3$ is the "fundamental" quantity, quantities such as the parallel Reynolds stress $\langle \delta U_\parallel \delta v_r \rangle$ must be extracted from the flux of magnetically weighted parallel momentum density. This requires subtracting off, or separating, the particle flux, which may produce unusual off-diagonal contributions to $\langle \delta U_\parallel \delta v_r \rangle$.

The physical origin of the ∇B-driven piece of the TurCo momentum pinch is easily revealed by considering the radial quasilinear turbulent flux of $m_i n U_\parallel / B^3$, the "magnetically weighted angular momentum" (MWA) density. Using $B \propto 1/R$, the MWA density maybe written as $m_i n U_\parallel R/B^2$ up to a constant, so that applying a straightforward quasilinear closure to Eq. (45) gives

$$\mathbf{\Pi}_{\mathrm{MWA}}^{\mathrm{Diff-QL}} \cdot \nabla \psi \equiv - \sum_{\mathbf{k}} (\mathrm{Re}\, \tau_{c\mathbf{k}} |\delta v_{r\mathbf{k}}|^2)\, \nabla\, (m_i n U_\parallel R/B^2) \cdot \nabla \psi = - \chi_{\mathrm{MWA}}^{\mathrm{QL}} \nabla (m_i n U_\parallel R/B^2) \cdot \nabla \psi. \quad (46)$$

Here, $\chi_{\mathrm{MWA}}^{\mathrm{QL}}$ is the quasilinear diffusivity for MWA. Note that

$$\chi_{\mathrm{MWA}} \equiv \left\langle \sum_{\mathbf{k}} \mathrm{Re}\, \tau_{c\mathbf{k}} |\delta v_{r\mathbf{k}}|^2 \right\rangle = \left\langle \left(\frac{c}{RB_\theta} \right)^2 \sum_{\mathbf{k}} \mathrm{Re}\, \tau_{c\mathbf{k}} \ell^2 |\delta \phi_{\mathbf{k}}|^2 \right\rangle,$$

and so is relatively insensitive to mode frequency and propagation direction. The flux $\mathbf{\Pi}_{\mathrm{MWA}}^{\mathrm{Diff-QL}}$ is driven by $\nabla(m_i n U_\parallel R/B^2) \cdot \nabla \psi$, and so has elements driven by ∇n and $\nabla(1/B)$, as well as $\nabla U_\parallel$. The $\nabla(1/B)$-driven piece is the nonthermodynamic-force-driven TurCo pinch. In particular, since

$$\nabla(m_i n U_\parallel R/B^2) = \nabla(L/B^2),$$

where $L \equiv m_i n U_\parallel R$ is the parallel angular momentum density, we have

$$\nabla(m_i n U_\parallel R/B^2) = (1/B^2) \nabla L + L \nabla (1/B^2). \quad (47)$$

This, in turn, implies

$$\mathbf{\Pi}_{\mathrm{MWA}}^{\mathrm{Diff-QL}} \cdot \nabla \psi = - \chi_{\mathrm{MWA}}^{\mathrm{QL}} \{ (1/B^2) \nabla L + L \nabla (1/B^2) \} \cdot \nabla \psi. \quad (48)$$

Hence, the transport evolution equation for MWA is, then, just

$$\frac{\partial}{\partial t}(L/B^2) + \nabla \cdot \mathbf{\Pi}_{\mathrm{MWA}}^{\mathrm{Diff-QL}} = 0. \quad (49)$$

Since B^2 is static, we have

$$\frac{1}{B^2} \frac{\partial}{\partial t} \langle L \rangle = - \langle \nabla \cdot \mathbf{\Pi}_{\mathrm{MWA}}^{\mathrm{Diff-QL}} \rangle = - \frac{1}{V'} \frac{\partial}{\partial \psi} [(V' \langle \mathbf{\Pi}_{\mathrm{MWA}}^{\mathrm{Diff-QL}} \cdot \nabla \psi \rangle] = \frac{1}{V'} \frac{\partial}{\partial \psi} [V' \langle \chi_{\mathrm{MWA}}^{\mathrm{QL}} \{ (1/B^2) \nabla L + L \nabla (1/B^2) \} \cdot \nabla \psi \rangle]. \quad (50)$$

Thus, we see that the total flux of parallel angular momentum density L consists of:

i) a diffusive piece, driven by ∇L,
ii) an off-diagonal, or convective piece, driven by ∇B.

Since $\nabla(1/B^2) > 0$, for outward-ballooning mode amplitude, this piece is indeed a pinch, and produces an inward flux of parallel angular momentum density. The pinch term described above corresponds to the ∇B-driven component of the TurCo flux of angular momentum.

The pinch of parallel angular momentum density described here is rather clearly of the TEP genre. This follows from the fact that it is ∇B-driven, and so *not* driven by a thermodynamic force. The ∇B-drive arises from the fact that proper symplectic nonlinear gyrokinetics[19] reveals that (to the lowest order in ϵ and $k_\parallel / k_\perp$), L/B^2 is locally advected, or "relaxed" and transported, so a homogenized state is one with $\partial / \partial \psi (L/B^2) = 0$, rather than with $(\partial / \partial \psi) L = 0$. The dynamics of homogenization and its relation to TEP pinches are discussed in Appendix B. Indeed, the condition of relaxation $(\partial / \partial \psi)(L/B^2) = 0$ defines a "canonical" profile of angular momentum density with gradient,

$$\left(\frac{\partial L}{\partial \psi} \right) \Big/ L = 2 \left(\frac{\partial B}{\partial \psi} \right) \Big/ B. \quad (51)$$

The canonical profile is the expected "end state" of the homogenization process, and so defines the limiting $(\nabla L)/L$ which may be "held" in the state of turbulent equipartition. Note too that the details of the turbulence dynamics do *not* enter the TEP theory, in that $\chi_{\mathrm{MWA}}^{\mathrm{QL}}$ is insensitive to the mode propagation direction etc., and depends *only* upon the correlation time and the spectrum of radial $\mathbf{E} \times \mathbf{B}$ velocities. It is always inward for outward ballooning mode structure.

Here, it is appropriate to compare and contrast the TEP theories for angular momentum and density. Both these theories yield pinches with roughly comparable magnitudes, which arise from the local advection and mixing of magnetically weighted quantities, namely L/B^2 in the case of angular momentum, and n/B in the case of a density transport model in a simple geometry (without consideration of magnetically trapped particles[58]) which is presented in Appendix B for an illustration of homogenization theory. More magnetic fusion-relevant TEP theories for density involve magnetically trapped electrons.[23–25] The dynamics for these is governed by bounce-kinetics in which parallel streaming averages out, and so is constrained by conservation of *two* adiabatic invariants, namely the magnetic moment μ, and the bounce action invariant J. Therefore, the commonality in their underlying physical mechanisms is obvious.

For completeness, we present a full expression of the TEP pinch originating from the homogenization of MWA. Writing the full expressions in Eq. (47), we have

$$\nabla(m_i n U_\parallel R/B^2) = (1/B^2) \nabla (m_i n R^2 \omega_\parallel) + m_i n R^2 \omega_\parallel \nabla (1/B^2).$$

Then,

$$\langle \mathbf{\Pi}_{\rm MWA}^{\rm Diff-QL}\cdot\nabla\psi\rangle = -\chi_{\rm MWA}^{\rm QL}\left\langle\left(\frac{RB_\theta}{B}\right)^2\frac{\partial}{\partial\psi}(m_i nR^2\omega_\parallel)\right\rangle + V_{\rm MWA}^{\rm TEP}\left\langle\frac{RB_\theta}{B^2}m_i nR^2\right\rangle\omega_\parallel, \tag{52}$$

where $\chi_{\rm MWA}^{\rm QL}$ is defined below Eq. (46), and

$$V_{\rm MWA}^{\rm TEP} \equiv -2\left\langle\sum_{\mathbf{k}} i\,{\rm Re}\,\tau_{c\mathbf{k}}\delta v_{r\mathbf{k}}^*\omega_{d-{\rm normak}}\frac{e\delta\phi_{\mathbf{k}}}{T_\perp}\right\rangle = 2\left\langle\frac{c}{RB_\theta}\sum_{\mathbf{k}}{\rm Re}\,\tau_{c\mathbf{k}}\omega_{d-{\rm normak}}\,\ell\,\frac{e}{T_\perp}|\delta\phi_{\mathbf{k}}|^2\right\rangle.$$

Here, we note that a contribution to $\omega_{d\perp\mathbf{k}}$ (which is the same as the $\omega_{d\parallel\mathbf{k}}$ in the low-β approximation) from the radial component of ∇B is $\omega_{d-{\rm normak}}\equiv(\ell cT_\perp/e_iB^2)(\partial B/\partial\psi)$. Now, it is quite obvious that $V_{\rm MWA}^{\rm TEP}$, derived above, is a part of $V_{\rm Ang}^{\rm TEP}$ in Eq. (36), showing they are from the same origin. Finally, there are additional contributions to the TurCo flux of angular momentum originating from the ion thermal effects, as discussed in other sections. This curvature driven thermoelectric (CTh) flux is ultimately driven by gradients in the thermodynamic variables (e.g., $\nabla T_\perp$ and $\nabla T_\parallel$), and the mode-dependency of the CTh flux is inevitable. Of course, the total turbulent convective (TurCo) flux of the parallel angular momentum,

$$\mathbf{\Pi}_{\rm Ang}^{\rm TurCo} = \mathbf{\Pi}_{\rm Ang}^{\rm TEP} + \mathbf{\Pi}_{\rm Ang}^{\rm CTh},$$

is an interesting and unusual combination of TEP and CTh contributions with different physics origins.

V. NONLINEAR GYROKINETIC EXPRESSION FOR TOROIDAL ANGULAR MOMENTUM DENSITY FLUX

The main goals of this section are to present the finite Larmor radius (FLR) version of the perturbed angular momentum response, and the turbulence driven mean radial flux of the perturbed angular momentum, and to show that we recover the results of Sec. III in the hydrodynamic limit. These were derived using a moment approach.

The ordering for this general formulation consists of

$$\omega/\Omega \sim e_i\delta\phi/T_i \sim \rho_i k_\parallel \sim \epsilon$$

and

$$k_\perp\rho_i \sim 1,$$

where ω and Ω are the characteristic fluctuation frequency and the ion cyclotron frequency, respectively; $k_\parallel$ and $k_\perp$ are the components of the wave vector in the parallel and perpendicular directions with respect to the magnetic field; ρ_i is the average ion gyroradius; $\delta\phi$ is the fluctuating electrostatic potential; and $\epsilon \ll 1$ is a small ordering parameter. As discussed in Sec. II, we take $U_0/v_{Ti}=O(\epsilon)<1$. A tokamak-specific ordering, $B_\theta/B\simeq rq/R\ll 1$, is implied since we take the parallel flow as an approximation to the toroidal flow. Here, r/R is the local inverse aspect ratio, and q is the magnetic safety factor. We start again from Eq. (6),

$$\frac{\partial\delta f}{\partial t} + \frac{d\mathbf{R}}{dt}\cdot\nabla\delta f + \frac{dv_\parallel}{dt}\frac{\partial\delta f}{\partial v_\parallel} = -\frac{d\mathbf{R}^{(1)}}{dt}\cdot\nabla F_0 - \frac{dv_\parallel^{(1)}}{dt}\frac{\partial F_0}{\partial v_\parallel}, \tag{53}$$

with

$$\frac{d\mathbf{R}^{(1)}}{dt} = c\frac{\mathbf{b}}{B^*}\times\nabla\langle\langle\delta\phi\rangle\rangle,$$

and

$$\frac{dv_\parallel^{(1)}}{dt} = -\frac{e_i}{m_i}\frac{\mathbf{B}^*}{B^*}\cdot\nabla\langle\langle\delta\phi\rangle\rangle.$$

We further simplify Eq. (53), ignoring terms involving $O(k_\parallel/k_\perp)$, $O(\rho_i/L_s)$, and write it in $\mathbf{k}$-space as

$$[-i(\omega_{\mathbf{k}}-\omega_{{\rm curv}\mathbf{k}}-\omega_{\nabla B\mathbf{k}}-v_\parallel k_\parallel)+\Delta\omega_{T\mathbf{k}}]\delta f_{\mathbf{k}} = \frac{ic\ell}{RB_\theta}\delta\phi_{\mathbf{k}}J_0\hat{\mathbf{e}}_\psi\cdot\nabla F_0 + i\frac{m_i}{e_i}k_\parallel\delta\phi_{\mathbf{k}}J_0\frac{\partial F_0}{\partial v_\parallel} + iv_\parallel\omega_{d\parallel\mathbf{k}}\frac{e\delta\phi_{\mathbf{k}}J_0}{T_\parallel}\frac{\partial F_0}{\partial v_\parallel}. \tag{54}$$

Here, $J_0=J_0(k_\perp v_\perp/(e_iB/m_ic))$, $\omega_{{\rm curv}\mathbf{k}}\equiv(cm_iv_\parallel^2/e_iB)\mathbf{b}\times(\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\mathbf{k}$, and $\omega_{\nabla B\mathbf{k}}\equiv(cm_i\mu/e_iB)\mathbf{b}\times\nabla B\cdot\mathbf{k}$. On the RHS, the first term is the $\mathbf{E}\times\mathbf{B}$ advection of F_0, the second term depends on the parallel acceleration, and the last term is the $\mathbf{B}^*$ modification to the parallel acceleration which is written in terms of the curvature drift of a thermal particle, $\omega_{d\parallel\mathbf{k}}$. Now using Eq. (54), we can calculate the angular momentum density perturbation:

$$\delta(nU_\parallel R) \equiv 2\pi\int d\mu dv_\parallel B^*\delta f_{\mathbf{k}}v_\parallel R,$$

as well as the nonlinear gyrokinetic expression for the mean turbulence driven radial flux of the angular momentum density carried by the fluctuating $\mathbf{E}\times\mathbf{B}$ velocity,

$$\langle\mathbf{\Pi}_{\rm Ang}^{\rm GK}\cdot\nabla\psi\rangle \equiv \left\langle 2\pi m_i\int d\mu dv_\parallel B^*\delta f v_\parallel R\frac{c}{B}\mathbf{b}\times\nabla\langle\langle\delta\phi\rangle\rangle\cdot\nabla\psi\right\rangle. \tag{55}$$

While the expression for $\langle\mathbf{\Pi}_{\rm Ang}^{\rm GK}\cdot\nabla\psi\rangle$ in Eq. (55) can be evaluated from nonlinear turbulent gyrokinetic simulations, a further explicit analytic evaluation of the kinetic expressions [including convoluted velocity-space integrals involving wave-particle resonances and finite Larmor radius (FLR) effects] is very complicated. Some general formulas are presented in Appendix C. We note that what we are calculating in this paper is the gyrocenter quantities, not the particle quantities. Therefore, we do not explicitly perform pullback transformations[59] from the gyrocenter quantities to the particle quantities, steps which are now routine in modern nonlinear gyrokinetic theories.[60] We also note that, including the FLR effects, the general gyrokinetic expression of the angular momentum flux in Eq. (55) includes an integration over μ which involves the μ-dependent $\delta f_{\mathbf{k}}$ and $\langle\langle\delta\phi_{\mathbf{k}}\rangle\rangle$. A simple decoupling of these terms is straightforward only in the long wavelength limit where $k_\perp\rho_i \ll 1$, i.e.,

$$\Pi^{GK}_{Ang}\cdot\nabla\psi \simeq m_i\delta(nU_\parallel R)\frac{c}{B}\mathbf{b}\times\nabla\delta\phi\cdot\nabla\psi$$

$$\simeq m_i cR\sum_{\mathbf{k}}\delta(nU_\parallel R)_{\mathbf{k}}\frac{\partial}{\partial\zeta}\delta\phi^*_{\mathbf{k}},$$

which is identical to that in Eq. (30). Approximate, but systematic ways to extend the decouplings of various hydrodynamic variables have been explored in the context of gyrofluid approaches.[61–63]

In passing, we discuss the gyrokinetic equivalent of the flux component

$$\Pi^{Turb}_{Geo} = (\mathbf{b}\times\nabla B)\left(\frac{\delta T_\perp + 3\delta T_\parallel}{B^2}\right)\delta(nU_\parallel R),$$

presented in Eq. (31), which is subdominant to Π_{Ang}. As discussed in Sec. III, one can consider the flux of the angular momentum density carried by the curvature drift and the grad-B drift to be

$$2\pi m_i\int d\mu dv_\parallel B^* f v_\parallel R\langle\langle \mathbf{v}_{curv}+\mathbf{v}_{\nabla B}\rangle\rangle.$$

Indeed, this expression can be deduced from an expression for the neoclassical momentum transport based on the Fokker-Planck equation.[50,64] In this paper, we study the turbulence driven angular momentum transport. For this, the quasilinear expression for the radial flux should involve the turbulence driven angular momentum density (i.e., a moment of δf) carried by the fluctuating curvature and grad-B velocities. Noting that $v_{curv}\propto v_\parallel^2$, and $v_{\nabla B}\propto \mu B$, the "turbulence-driven fluctuating" curvature and grad-B velocities should involve the temperature fluctuations. Thus, we can identify the gyrokinetic expression for the flux of the angular momentum density carried by the curvature drift and grad-B drift as

$$\langle\Pi^{GK}_{Geo}\cdot\nabla\psi\rangle = \left\langle 2\pi m_i\int d\mu dv_\parallel\left[B^*\delta f v_\parallel R\left\langle\left\langle\frac{v_\parallel^2-3v_{T\parallel}^2}{e_iB}\mathbf{b}\times(\mathbf{b}\cdot\nabla)\mathbf{b}+\frac{\mu B - v_{T\perp}^2}{e_iB^2}\mathbf{b}\times\nabla B\right\rangle\right\rangle\right]\cdot\nabla\psi\right\rangle. \quad (56)$$

As discussed in Sec. III, this term is on the order of $o(\omega_d/\omega)$ smaller than Π^{TurCo}_{Ang}, and we do not pursue reduction of these terms further in this paper. An evaluation of this flux from turbulent nonlinear gyrokinetic simulations will be more complicated than that for Π^{GK}_{Ang}. However, the turbulent contribution to this flux expression should be revisited when fully nonlinear simulations, including both neoclassical and turbulent effects, are attempted.

Now returning to the mean turbulence driven radial flux of the angular momentum density carried by the fluctuating $\mathbf{E}\times\mathbf{B}$ velocity, and using the expression for $\delta f_{\mathbf{k}}$ in Eq. (54), we can write a more explicit theoretical expression for Π^{GK}_{Ang} as

$$\Pi^{GK}_{Ang} \equiv 2\pi m_i\sum_{\mathbf{k}}\int d\mu dv_\parallel B^*[-i(\omega_{\mathbf{k}}-\omega_{curv\mathbf{k}}-\omega_{\nabla B\mathbf{k}}-v_\parallel k_\parallel)+\Delta\omega_{T\mathbf{k}}]^{-1}\left\{\frac{ic\ell}{RB_\theta}J_0\delta\phi_{\mathbf{k}}\hat{\mathbf{e}}_\psi\cdot\nabla F_0 + i\left(\frac{m_i}{e_i}k_\parallel J_0\delta\phi_{\mathbf{k}}+v_\parallel\omega_{d\parallel\mathbf{k}}\frac{e\delta\phi_{\mathbf{k}}J_0}{T_\parallel}\right)\frac{\partial F_0}{\partial v_\parallel}\right\}\times v_\parallel R\left(\frac{ic\ell}{RB_\theta}\right)J_0\delta\phi_{-\mathbf{k}}. \quad (57)$$

As is well known, when $\Delta\omega_T > \omega,\omega_{curv},\omega_{\nabla B},k_\parallel v_\parallel$, for significant nonlinear frequency broadening, strong turbulence theory applies. But when $\Delta\omega_T \ll \omega,\omega_{curv},\omega_{\nabla B},k_\parallel v_\parallel$, quasilinear scaling applies, and turbulent flux scales with the fluctuation intensity. In the case that $\Delta\omega_T$ is negligible, $\tau_{c\mathbf{k}}$ must arise from resonant wave-particle interaction restricted by $\delta(\omega_{\mathbf{k}}-\omega_{curv\mathbf{k}}-\omega_{\nabla B\mathbf{k}}-k_\parallel v_\parallel)$. General formalisms for transport in the quasilinear regime focusing on the roles of the resonant wave-particle interactions have been presented for drift waves in cylindrical geometry[65] and for generic low frequency fluctuations in a dipole geometry.[66] Neither of these works, nor another which addresses the neoclassical transport matrix[67] has predicted the possibility of a turbulent convective flux of toroidal angular momentum density as discussed in this paper.

To make a direct connection to the results from the moment approach in Sec. III, we make the following hydrodynamic limit approximation which is slightly different from the usual one. This somewhat unusual expansion allows us to relate and connect terms which emerge from this kinetic calculation to the various contributions to the angular momentum flux we obtain using the fluid theory in Sec. III. The hydrodynamic expansion is based upon the following disparities in spatio-temporal scales,

$$\omega,\Delta\omega_T \gg \omega_{curv},\omega_{\nabla B},k_\parallel v_\parallel$$

and

$$k_\perp\rho_i \ll 1.$$

Guided by Eq. (32), we expand the renormalized propagator in terms of the ratio,

$$\frac{\omega_{curv}-3\omega_{d\parallel}+\omega_{\nabla B}-\omega_{d\perp}+k_\parallel v_\parallel}{\omega-3\omega_{d\parallel}-\omega_{d\perp}+i\Delta\omega_T}.$$

Note that ω_{curv} and $\omega_{\nabla B}$ are velocity-dependent, and that $3\omega_{d\parallel}$ and $\omega_{d\perp}$ are their appropriate thermal average values. Therefore, the denominator in this expansion is independent of the particle velocity. In the limit $k_\parallel v_\parallel \to 0$, the numerator of this expansion is $\omega_{curv}-3\omega_{d\parallel}+\omega_{\nabla B}-\omega_{d\perp}=\omega_{d\parallel}(v_\parallel^2/v_{T_\parallel}^2-3)+\omega_{d\perp}(\mu B/v_{T_\perp}^2-1)$. Here, the subscript $\mathbf{k}$ is understood. For simplicity, we neglect the $k_\parallel v_\parallel$ term in the propagator hereafter, but we present some kinetic results in Appendix C. The $k_\parallel v_\parallel$ term is related to the acoustic dynamics along the magnetic field, and plays an important role in theories in simple geometry.[15,52] The kinetic expression for the $k_\parallel$-dependent angular momentum flux has been given in Ref. 15, and is not repeated here. Then, focusing on the perpendicular dynamics, the inversion of the renormalized propagator can be approximated by

$$[-i(\omega-\omega_{\text{curv}}-\omega_{\nabla B})+\Delta\omega_T]^{-1} \simeq [-i\ (\omega-3\omega_{d\parallel}-\omega_{d\perp})+\Delta\omega_T)]^{-1} \times\left[1+\frac{\omega_{d\parallel}(v_\parallel^2/v_{T\parallel}^2-3)+\omega_{d\perp}(\mu B/v_{T\perp}^2-1)}{\omega-3\omega_{d\parallel}-\omega_{d\perp}+i\Delta\omega_T}\right].$$

In this limit, we can identify the terms contributing to the diffusive angular momentum density flux, $\mathbf{\Pi}_{\text{Ang}}^{\text{GK,Diff}}$, the TEP angular momentum density flux, $\mathbf{\Pi}_{\text{Ang}}^{\text{GK,TEP}}$, and the CTh angular momentum density flux, $\mathbf{\Pi}_{\text{Ang}}^{\text{GK,CTh}}$, respectively. Using the leading order renormalized propagator and considering the relaxation of ∇F_0 due to the fluctuating $\mathbf{E}\times\mathbf{B}$ velocity, we obtain

$$\mathbf{\Pi}_{\text{Ang}}^{\text{GK,0}}=-2\pi m_i\sum_{\mathbf{k}}\int d\mu dv_\parallel B^*\ \text{Re}[-i(\omega_{\mathbf{k}}-3\omega_{d\parallel}-\omega_{d\perp}) +\Delta\omega_{T\mathbf{k}}]^{-1}v_\parallel R\left(\frac{c\ell}{RB_\theta}\right)^2 J_0^2|\delta\phi_{\mathbf{k}}|^2\nabla F_0. \quad (58)$$

Also, from the leading order term in the renormalized propagator and the ($\mathbf{B}^*$-related) curvature drift correction to the parallel acceleration (which relaxes the $v_\parallel$-gradient of F_0), we obtain

$$\mathbf{\Pi}_{\text{Ang}}^{\text{GK,TEP}}\equiv-2\pi m_i\sum_{\mathbf{k}}\int d\mu dv_\parallel B^*\ \text{Re}[-i(\omega_{\mathbf{k}}-3\omega_{d\parallel}-\omega_{d\perp}) +\Delta\omega_{T\mathbf{k}}]^{-1}\left[v_\parallel\omega_{d\parallel\mathbf{k}}\frac{e}{T_\parallel}\frac{\partial F_0}{\partial v_\parallel}\right]v_\parallel R\frac{c\ell}{RB_\theta}J_0^2|\delta\phi_{\mathbf{k}}|^2\hat{\mathbf{e}}_\psi. \quad (59)$$

Finally, from the first order ($o(\omega_d/\omega)$) correction term to the renormalized propagator in the hydrodynamic expansion, and the relaxation of ∇F_0 due to the fluctuating $\mathbf{E}\times\mathbf{B}$ velocity, we obtain

$$\mathbf{\Pi}_{\text{Ang}}^{\text{GK,1}}=-2\pi m_i\sum_{\mathbf{k}}\int d\mu dv_\parallel B^*\ \text{Re}\left[\{-i(\omega_{\mathbf{k}}-3\omega_{d\parallel}-\omega_{d\perp}) +\Delta\omega_{T\mathbf{k}}\}^{-1}\frac{\omega_{d\parallel}(v_\parallel^2/v_{T\parallel}^2-3)+\omega_{d\perp}(\mu B/v_{T\perp}^2-1)}{\omega-3\omega_{d\parallel}-\omega_{d\perp}+i\Delta\omega_T}\right] \times v_\parallel R\left(\frac{c\ell}{RB_\theta}\right)^2 J_0^2|\delta\phi_{\mathbf{k}}|^2\nabla F_0. \quad (60)$$

For the explicit calculation of the expressions in Eqs. (58)–(60), the gradients of F_0 in the phase-space should be evaluated using a specific choice of F_0. Assuming a shifted (in $v_\parallel$) local Maxwellian F_0, we have

$$\nabla\ln F_0=\nabla\ln n_0+\frac{1}{2}\left(\frac{(v_\parallel-U_0)^2}{v_{Ti}^2}-1\right)\nabla\ln T_\parallel +\left(\frac{\mu B}{T_\perp}-1\right)\nabla\ln T_\perp+\frac{m_i}{T_\parallel}(v_\parallel-U_0)\nabla U_0-\frac{\mu}{T_\perp}\nabla B, \quad (61)$$

and

$$\frac{\partial\ln F_0}{\partial v_\parallel}=-\frac{m_i}{T_\parallel}(v_\parallel-U_0).$$

Here, all the derivatives are taken in $(\mathbf{R},\mu,v_\parallel)$ space. We note that many integrals vanish due to the odd parity of the integrands in $v_\parallel-U_0$. For this choice of F_0, only the $(m_i/T_\parallel)(v_\parallel-U_0)\nabla U_0$ term, the $\nabla\ln n_0$ term, and the $(\mu/T_\perp)\nabla B$ term contribute to $\mathbf{\Pi}_{\text{Ang}}^{\text{GK,0}}$. Note that two thermodynamic driving terms ∇n_0 and ∇U_0, and a geometric correction $\nabla B\propto-\nabla R$ necessary for the angular momentum density, can be combined into $\Gamma_0(b_i)\nabla(n_0U_0R)$, after velocity-space integrations. Here, $\Gamma_n(b_i)\equiv I_n(b_i)\exp(-b_i)$, the I_n's are the modified Bessel functions, and $b_i=k_\perp^2\rho_i^2$. Due to additional μ dependence, the ∇B-driven term produces a FLR residual contribution $\propto b_i(\Gamma_1-\Gamma_0)\nabla R$. As we will explain shortly, from $\mathbf{\Pi}_{\text{Ang}}^{\text{GK,1}}$, we obtain FLR residual terms driven by ∇U_0 and $\nabla\ln n_0$ which are also proportional to $\propto b_i(\Gamma_1-\Gamma_0)$. So, within this hydrodynamic limit, the full Larmor radius version of the diffusive flux $\mathbf{\Pi}_{\text{Ang}}^{\text{GK,Diff}}$ has a relatively compact form which is

$$\langle\mathbf{\Pi}_{\text{Ang}}^{\text{GK,Diff}}\cdot\nabla\psi\rangle=-\left\langle\sum_{\mathbf{k}}\left[\text{Re}\ \tau_{c\mathbf{k}}\left(\frac{c\ell}{RB_\theta}\right)^2 \times(\Gamma_0+b_i(\Gamma_1-\Gamma_0)) \times|\delta\phi_{\mathbf{k}}|^2\nabla(m_in_0U_0R)\cdot\nabla\psi\right.\right\rangle. \quad (62)$$

Then, as in Sec. III, from

$$\langle\mathbf{\Pi}_{\text{Ang}}^{\text{GK,Diff}}\cdot\nabla\psi\rangle\equiv-\chi_{\text{Ang}}^{\text{GK}}\left\langle(RB_\theta)^2\frac{\partial}{\partial\psi}(m_in_0R^2\omega_\parallel)\right\rangle,$$

we can define the flux-surface-averaged angular momentum density diffusivity,

$$\chi_{\text{Ang}}^{\text{GK}}=\left\langle\left(\frac{c}{RB_\theta}\right)^2\sum_{\mathbf{k}}\text{Re}\ \tau_{c\mathbf{k}}\ell^2(\Gamma_0+b_i(\Gamma_1-\Gamma_0))|\delta\phi_{\mathbf{k}}|^2\right\rangle. \quad (63)$$

$\mathbf{\Pi}_{\text{Ang}}^{\text{GK,TEP}}$ is relatively insensitive to details of the hydrodynamic expansion, and to the choice of F_0. As one can check via an integration by parts, the TEP pinch can be easily evaluated by assuming that F_0 is an even function of $(v_\parallel-U_0)$ (i.e., without using a specific F_0 explicitly). In a collisionless Hamiltonian system, only an F_0 which is a function of the constants of the motion (μ,L_ϕ,E) alone, exactly satisfies the zeroth order nonlinear gyrokinetic equation. Here, E is the single particle energy in the absence of a time-dependent electromagnetic field. Use of the usual choice of a shifted Maxwellian for F_0 typically causes an error on the order of $v_\parallel/\Omega_\theta L_\perp$, with $\Omega_\theta=e_iB_\theta/m_ic$, and a characteristic gradient length in the perpendicular direction $L_\perp$. Note that those errors originate from using the radial coordinate ψ in lieu of the canonical angular momentum $L_\phi \simeq (e_i/c)\psi + m_i v_\| R$, as the argument of n_0, T_0, and U_0. On the other hand, $\mathbf{\Pi}_{\mathrm{Ang}}^{\mathrm{GK,TEP}}$ is driven by the gradient in $v_\|$-space, and is free from the aforementioned error. The "TEP" mechanism relies almost entirely on the single particle guiding center dynamics and is relatively insensitive to the choice of F_0. For the gyrokinetic expression for the TEP flux, we have

$$\langle \mathbf{\Pi}_{\mathrm{Ang}}^{\mathrm{GK,TEP}} \cdot \nabla \psi \rangle = -2\left\langle \sum_{\mathbf{k}} \mathrm{Re}\, \tau_{c\mathbf{k}} \omega_{d\|\mathbf{k}} \frac{e}{T_\|} \frac{c\ell}{RB_\theta} \Gamma_0 |\delta\phi_{\mathbf{k}}|^2 (m_i n_0 R^2) \omega_\| R B_\theta \right\rangle. \tag{64}$$

Then, as in Sec. III, from

$$\langle \mathbf{\Pi}_{\mathrm{Ang}}^{\mathrm{GK,TEP}} \cdot \nabla \psi \rangle \equiv \langle m_i n_0 R^3 B_\theta \rangle \omega_\| V_{\mathrm{Ang}}^{\mathrm{GK,TEP}},$$

we can define the flux-surface-averaged "TEP angular momentum density pinch,"

$$V_{\mathrm{Ang}}^{\mathrm{GK,TEP}} = 2\left\langle \frac{c}{RB_\theta} \sum_{\mathbf{k}} \mathrm{Re}\, \tau_{c\mathbf{k}} \omega_{d\|\mathbf{k}}\, \ell\, \frac{e}{T_\|} \Gamma_0 |\delta\phi_{\mathbf{k}}|^2 \right\rangle = -2\left\langle \frac{1}{R}\left(\frac{c}{RB_\theta}\right)^2 \sum_{\mathbf{k}} \mathrm{Re}\, \tau_{c\mathbf{k}} \ell^2 \frac{\omega_{d\|\mathbf{k}}(\theta)}{\omega_{d\|\mathbf{k}}(0)} \Gamma_0 |\delta\phi_{\mathbf{k}}|^2 \right\rangle. \tag{65}$$

On the other hand, the temperature-gradient-related terms contribute to $\mathbf{\Pi}_{\mathrm{Ang}}^{\mathrm{GK,CTh}}$. Since the velocity dependence of the renormalized propagator has been approximated using a particular version of the hydrodynamic expansion, the expression for $\mathbf{\Pi}_{\mathrm{Ang}}^{\mathrm{GK,1}}$ is less robust for a particular choice of the theoretical framework, as demonstrated further in Appendix C. After evaluating the velocity space integral in Eq. (60), we note that the terms driven by the temperature gradients $\nabla T_\|$ and $\nabla T_\perp$ in Eq. (61) can be identified as $\mathbf{\Pi}_{\mathrm{Ang}}^{\mathrm{GK,CTh}}$. Other FLR residual terms driven by ∇U_0 and $\nabla \ln n_0$, which are also proportional to $b_i(\Gamma_1 - \Gamma_0)$, can be absorbed into the diffusive flux $\mathbf{\Pi}_{\mathrm{Ang}}^{\mathrm{GK,Diff}}$. Thus, with its dependence on $\omega_{d\|}$ and $\omega_{d\perp}$, this flux can be characterized as the CTh (curvature driven thermoelectric) flux. For the gyrokinetic expression for the CTh flux, we have

$$\langle \mathbf{\Pi}_{\mathrm{Ang}}^{\mathrm{GK,CTh}} \cdot \nabla \psi \rangle = -\left\langle \sum_{\mathbf{k}} \mathrm{Re}[\{-i(\omega - 3\omega_{d\|} - \omega_{d\perp}) + \Delta\omega_T)\}^{-1}\{\omega - 3\omega_{d\|} - \omega_{d\perp} + i\Delta\omega_T\}^{-1}] \times [\Omega_d \Omega_{*T}](\mathrm{FLR}) \times \left(\frac{c\ell}{RB_\theta}\right) \frac{e}{T_i} |\delta\phi_{\mathbf{k}}|^2 (m_i n_0 R^2) \omega_\| R B_\theta \right\rangle, \tag{66}$$

where $[\Omega_d \Omega_{*T}](\mathrm{FLR}) \equiv 3\omega_{d\|\mathbf{k}} \omega_{*T\|} \Gamma_0 + \omega_{d\perp\mathbf{k}} \omega_{*T\perp} \{(1-2b_i + 2b_i^2)\Gamma_0 + (b_i - 2b_i^2)\Gamma_1\}$. Then, as in Sec. III, using

$$\langle \mathbf{\Pi}_{\mathrm{Ang}}^{\mathrm{GK,CTh}} \cdot \nabla \psi \rangle \equiv \langle m_i n_0 R^3 B_\theta \rangle \omega_\| V_{\mathrm{Ang}}^{\mathrm{GK,CTh}},$$

we can define the flux-surface-averaged "CTh angular momentum density pinch," which is

$$V_{\mathrm{Ang}}^{\mathrm{GK,CTh}} = -\left\langle \sum_{\mathbf{k}} \mathrm{Re}[\{-\ i(\omega - 3\omega_{d\|} - \omega_{d\perp}) + \Delta\omega_T)\}^{-1}\{\omega - 3\omega_{d\|} - \omega_{d\perp} + i\Delta\omega_T\}^{-1}][\Omega_d \Omega_{*T}](\mathrm{FLR}) \times \left(\frac{c\ell}{RB_\theta}\right) \frac{e}{T_i} |\delta\phi_{\mathbf{k}}|^2 \right\rangle, \tag{67}$$

where $\omega_{d\|} < 0$, and $\omega_{d\perp} < 0$ at the low-B field side midplane. The total turbulent convection (TurCo) velocity is, again, given by

$$V_{\mathrm{Ang}}^{\mathrm{GK,TurCo}} = V_{\mathrm{Ang}}^{\mathrm{GK,TEP}} + V_{\mathrm{Ang}}^{\mathrm{GK,CTh}},$$

with the TEP contribution and CTh contribution given by Eq. (65) and Eq. (67), respectively.

VI. CONCLUSIONS

In this paper, we presented the nonlinear gyrokinetic theory of the toroidal momentum pinch. We develop the theory from a symplectic gyrokinetic equation in toroidal geometry,[19] which conserves phase space density and energy. The principal results of this paper are:

(i) The total flux of toroidal angular momentum density is calculated. This is shown to consist of three pieces, namely, the now-familiar diffusive flux,[2] a novel turbulent convective flux with velocity $V_{\mathrm{Ang}}^{\mathrm{TurCo}}$, and an off-diagonal flux produced by acoustic perturbations in the presence of broken $x \to -x$ symmetry, as discussed in Ref. 15.

(ii) The novel convective velocity $V_{\mathrm{Ang}}^{\mathrm{TurCo}}$ is shown in turn to consist of two distinctive components produced by two distinctive processes. $V_{\mathrm{Ang}}^{\mathrm{TurCo}}$ consists of a $\nabla(1/B)$-driven turbulent equipartition (TEP) convective velocity (*not* produced by a thermodynamic force) and a curvature driven thermal (CTh) convective velocity (produced by ∇T_i, a thermodynamic force). The TEP component of $V_{\mathrm{Ang}}^{\mathrm{TurCo}}$ arises from electrostatic acceleration along curved field lines ($\sim -m_i c v_\| \nabla \times \mathbf{b} \cdot \nabla \delta\phi$) and its resulting contribution to the parallel Reynolds stress $\langle \delta v_r \delta v_\| \rangle$, which coexists with the usual parallel acceleration in toroidal geometry. Both components of $V_{\mathrm{Ang}}^{\mathrm{TurCo}}$ require symmetry breaking via ballooning mode structure to exist, and will vanish for flute-like fluctuations with $\delta\phi$=const on a flux surface.

(iii) The $\nabla(1/B)$-driven TEP piece of $V_{\mathrm{Ang}}^{\mathrm{TurCo}}$ is shown to arise from the fact that, in a low-β tokamak equilibrium, $B^2 \mathbf{u}_E = c\mathbf{B} \times \nabla \delta\phi$ is approximately incompressible, so that the magnetically weighted angular momentum density ($m_i n U_\| / B^3 \propto m_i n U_\| R / B^2 = L/B^2$, since $B \propto 1/R$) is locally advected by fluctuating $\mathbf{E} \times \mathbf{B}$ velocities, to the lowest order in $O(a/R)$. As a consequence L/B^2 is mixed or homogenized, so that $(\partial/\partial\psi)(L/B^2) \to 0$. Thus, the $\nabla(1/B)$-driven $V_{\mathrm{Ang}}^{\mathrm{TurCo}}$ pinch is seen to be of the turbulent equipartition variety, and is *not* driven by a thermodynamic force. Typically, $V_{\mathrm{Ang}}^{\mathrm{TEP}}$ is given by

$$V_{\mathrm{Ang}}^{\mathrm{TEP}} \simeq -\frac{2}{R_0} \chi_{\mathrm{Ang}},$$

for outward ballooning fluctuations (peaked at the low-B side). Here, χ_{Ang} is the angular momentum density diffusivity, similar to χ_ϕ. The TurCo TEP pinch, $V_{\rm Ang}^{\rm TEP}$, is insensitive to mode phase velocity.

(iv) On the other hand, the curvature driven thermal (CTh) flux is shown to be ∇T_i-driven, and so is of the ion thermoelectric variety. Typically,

$$V_{\rm Ang}^{\rm CTh} \simeq -\frac{4G^{\rm Th}}{R_0}\chi_{\rm Ang},$$

where $G^{\rm Th} \simeq \delta T_i / e\delta\phi$. Unlike $V_{\rm Ang}^{\rm TEP}$ which is inward regardless of microinstability details, $V_{\rm Ang}^{\rm CTh}$ depends on the direction of mode propagation. Thus, roughly speaking, for fluctuations propagating in the electron diamagnetic direction, $G^{\rm Th}$ is definitely positive, making $V_{\rm Ang}^{\rm CTh}$ inward for outward ballooning fluctuations. For fluctuations propagating in the ion diamagnetic direction, $G^{\rm Th}$ can be negative (but not always), and $V_{\rm Ang}^{\rm CTh}$ can be outward for outward ballooning fluctuations. We emphasize, though, that numerical calculations are usually required to determine the net direction or sign of $V_{\rm Ang}^{\rm CTh}$. The trends in the various contributions to $V_{\rm Ang}^{\rm TurCo}$ are summarized in Table III.

(v) The basic implications for tokamak experiments have been outlined. Since both $V_{\rm Ang}^{\rm TEP}$ and $V_{\rm Ang}^{\rm CTh}$ are inward for fluctuations propagating in the electron diamagnetic direction, we expect the total convective pinch velocity, $V_{\rm Ang}^{\rm TurCo}=V_{\rm Ang}^{\rm TEP}+V_{\rm Ang}^{\rm CTh}$, to be *inward* for TEM-dominated turbulence, which is expected for Ohmic and electron-heated plasmas. On the other hand, for discharges where transport is determined by ITG-dominated turbulence, $V_{\rm Ang}^{\rm CTh}$ can sometimes be outward, while $V_{\rm Ang}^{\rm TEP}$ is always inward, making the net sign of $V_{\rm Ang}^{\rm TurCo}$ a question of detail. Note, however, that the off-diagonal piece of $\langle \delta v_r \delta U_\parallel \rangle$ produced by the synergism between the parallel acceleration by $\nabla_\parallel \delta\phi$ and $x \to -x$ symmetry breaking by $\mathbf{E}\times\mathbf{B}$ shear is usually inward for ITG-driven turbulence. Thus, the toroidal mechanism for the TurCo pinch nicely complements that mechanism, and can help explain (via an inward pinch of momentum) the appearance of spontaneous or intrinsic rotation in electron heated plasmas. However, some synergism between the TurCo pinch and the electric field driven residual stress of Ref. 15 is probably necessary to explain both the profile structure and the Rice scaling of intrinsic rotation exhibited in Ref. 17.

Several other comments are in order here. First, this calculation is a good example of how consideration of the subtleties of modern gyrokinetics can lead one to identifying a novel physics effect, as well as improve the treatment of familiar ones. Indeed, this is likely the first significant example of such a discovery. Second, it should be clear, that this calculation is in the spirit of quasilinear theory, and focuses on evaluating the momentum flux given an absolutely minimal characterization of the turbulence. In particular, effects of mode-mode coupling, turbulence spreading, and nonlinear wave-particle interaction—all of which may contribute to nondiffusive momentum transport—are not addressed here. Third, the calculation discussed here is primarily concerned with calculating the flux of magnetically weighted angular momentum density $m_i n U_\parallel / B^3$. Indeed, a major result of this paper is the identification of that quantity as one which is (approximately) locally conserved and homogenized. However, experiments often are mostly concerned with the parallel Reynolds stress $\langle \delta v_r \delta U_\parallel \rangle$, and thus some care is required in subtracting off the contribution from particle flux $U_0 \langle \delta v_r \delta n \rangle / B^3$ from $\langle \delta v_r \delta L / B^2 \rangle$. This is discussed in Appendix A. Also, we note that the treatment here applies only to electrostatic microturbulence at low β.

Finally, we note that, like virtually all theories of toroidal momentum transport and spontaneous/intrinsic rotation, this paper does not address either the role of perpendicular flows in toroidal momentum transport or the dynamics of poloidal momentum transport. Both of these can be quite important, since experimental evidence for non-neoclassical poloidal flows is accumulating.[68,69] Noting the richness of turbulence-driven flow physics,[20] we note that a proper gyrokinetic treatment of this problem requires a lengthy calculation along the lines of Ref. 70. This calculation will be presented in a future paper.

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APPENDIX A: TREATMENT OF PARTICLE FLUX

The main subject of Appendix A is to discuss the relationships between the quantities derived in the main text and those commonly used in the transport analyses of experimental data. We also discuss the effects of particle flux on rotation evolution. While it is most natural to study the evolution of the angular momentum density, $nU_\phi R$, for theoretical studies of the toroidal momentum transport, and for analysis of perturbative momentum experiments,[1,12,13] what is measured and estimated from experiments is the toroidal velocity $U_\phi = \omega_\phi R$. In an analysis of experimental data,[1] the "toroidal (linear) momentum diffusivity," $\chi_{{\rm Mom},\phi}$ was defined from

$$\begin{aligned}\mathbf{\Pi}_{\rm Mom} &\equiv -\chi_{{\rm Mom},\phi} n_0 m_i \nabla U_\phi + m_i U_\phi \mathbf{\Gamma}_{\rm ion} \\ &\equiv -\chi_{{\rm Mom},\phi}^{\rm eff} n_0 m_i \nabla U_\phi. \end{aligned} \tag{A1}$$

Here, $\mathbf{\Pi}_{\rm Mom}$ is the total radial flux of the (linear) toroidal momentum density $\delta(nU_\phi)\delta v_r$, $\mathbf{\Gamma}_{\rm ion}$ is the ion particle flux. Note that the TurCo momentum pinch terms discussed in this paper are *not* included in this relation. In a simpler characterization, $\chi_{{\rm Mom},\phi}^{\rm eff}$ includes contributions from both diffusive momentum flux and an apparent "convective" momentum flux which comes from the ion particle flux. Thus, even before getting into the issue of the possible TurCo momentum pinch (the main contribution of this paper), we recognize the importance of a proper treatment of particle transport in the momentum transport studies. Of course, quasineutrality requires $\mathbf{\Gamma}_{\rm ion}=\mathbf{\Gamma}_{\rm electron}=\mathbf{\Gamma}_{\rm ptl}$, and $\mathbf{\Gamma}_{\rm ptl}=-D_{\rm ptl}\nabla n+V_{\rm ptl}n$ is a typical characterization. Hence, the convective particle pinch can result in an inward pinch of toroidal momentum.

One might think the influence of particle transport on the characterization of momentum transport can be avoided by calculating the radial flux of rotation (i.e., $U_\parallel$ without a density multiplier) directly. However, this is not, in general, true, since the dynamics of $U_\parallel$ will be coupled to that of density n even more strongly than the dynamics of $nU_\parallel$ is. We think with the possible exception of "pure" ITG turbulence with no particle flux (i.e., due to Boltzmann electrons), the calculation of the radial flux of rotation from the gyrokinetic or moment approach will be more complex as compared to that of our approach in the main text (i.e., calculating the radial flux of the momentum density $nU_\parallel$). We claim that from our calculation of the total radial flux of the (linear) toroidal momentum density, $\mathbf{\Pi}_{\rm Mom}$, in the main text, one should define the momentum diffusivity and various pinch velocities as follows:

$$\mathbf{\Pi}_{\rm Mom} \equiv -\chi_{{\rm Mom},\phi} n_0 m_i \nabla U_\phi + \mathbf{V}_{\rm Mom}^{\rm TurCo} n_0 m_i U_\phi + m_i U_\phi \mathbf{\Gamma}_{\rm ptl}, \tag{A2}$$

where $\mathbf{V}_{\rm Mom}^{\rm TurCo}$ is the "turbulent convective" radial pinch velocity of the momentum density. When one calculates the evolution of the flow using the continuity equation, a contribution to $(\partial/\partial t)n$ coming from $\nabla\cdot\mathbf{\Gamma}_{\rm ptl}$ appears. Sometimes one neglects the influence of the particle flux on the flow evolution, assuming a negligible particle source at the core. However, in general, from Eq. (A2), it is obvious that $\mathbf{\Gamma}_{\rm ptl}$ can manifest itself as an apparent "velocity pinch" if one does not elaborate on the particle flux in studying the momentum transport. Since the particle flux can manifest itself as an apparent momentum pinch, it is instructive to compare a typical particle pinch velocity to the $V_{\rm Ang}^{\rm TEP}$ in Eq. (37). Since the magnitude of the particle pinch varies considerably depending on plasma conditions, it makes more theoretical sense to compare the particle pinch and the momentum pinch from similar physical origins. Therefore, we compare $V_{\rm Ang}^{\rm TEP}$ with $V_{\rm ptl}^{\rm TEP}$ from Ref. 25 which obtained

$$V_{\rm ptl}^{\rm TEP} \simeq -2(D_{\rm ptl}/R_0)\left(\frac{1}{4}+\frac{2\hat{s}}{3}\right).$$

Thus, we see that, in normalized form for comparison,

$$|V_{\rm Ang}^{\rm TEP}/V_{\rm ptl}^{\rm TEP}| \sim (\chi_{\rm Ang}/D_{\rm ptl})\left(F_{\rm balloon}\Big/\left(\frac{1}{4}+\frac{2\hat{s}}{3}\right)\right).$$

While $\chi_{\rm Ang}>D_{\rm ptl}$ typically, they are roughly of the same order. Furthermore, with a contribution to $F_{\rm balloon}$ coming from $F_{\rm geo}$ which depends on $\hat{s}$, the ratio $F_{\rm balloon}/\left(\frac{1}{4}+\frac{2\hat{s}}{3}\right)$ is typically on the order of unity. A notable difference is the fact that trapped electrons, for which the response is bounce-averaged, carry the particle pinch in particle TEP theories, but circulating ions carry the momentum pinch in our theory. Thus, the particle transport contribution should be kept in mind, when one studies momentum transport. Note also that possible confusion from considering the ratio $|V_{\rm Mom}^{\rm TEP}/V_{\rm ptl}^{\rm TEP}|$ for reversed magnetic shear plasmas is unfounded, since the turbulence is so weak under these conditions that the assumption of homogenization, which is generic to TEP models, is dubious.

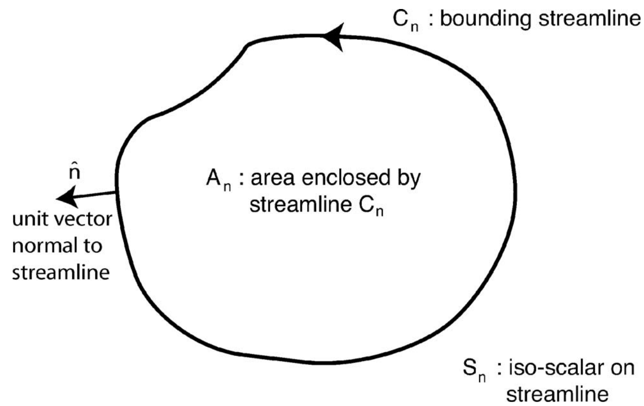

FIG. 1. Geometry nomenclature for homogenization.

APPENDIX B: DYNAMICS OF TURBULENT EQUIPARTITION FLUXES AND HOMOGENIZATION

In Appendix B, we review the physics of TEP fluxes in the light of homogenization theory. The aim here is to elucidate the fundamentals of TEP theory using ideas relevant to homogenization and transport of potential vorticity and scalar concentration in 2D incompressible flows. The latter provide useful, unifying principles within which to consider a variety of problems involving mixing, transport, and relaxation. In particular, turbulent equipartition (TEP) pinches emerge as effects which limit complete homogenization due to (effectively) compressible dynamics.

Homogenization theory, derived from the Prandtl-Batchelor theorem,[71] is concerned with the mixing of a scalar quantity within a region bounded by a closed streamline in a 2D incompressible flow. The basic equation of the homogenization problem is

$$\frac{\partial}{\partial t}S+\nabla\phi\times\hat{\mathbf{z}}\cdot\nabla S=\nu\nabla^2 S, \tag{B1}$$

with $\mathbf{v}=\nabla\phi\times\hat{\mathbf{z}}$ satisfying $\nabla\cdot\mathbf{v}=0$. Here $S=\nabla^2\phi$ (vorticity) for a 2D fluid, $S=-\beta y+\nabla^2\phi$ (potential vorticity) for a geostrophic fluid, $S=\ln n_0+\phi-\nabla^2\phi$ for 2D drift wave turbulence, and $S=A$ (magnetic potential or other scalar field) for scalar evolution. We will show that ultimately $S\to$const within a closed, bounding streamline C_0. We consider a particular closed streamline C_n within C_0.

Homogenization requires that the small scale dissipation be diffusive $(\sim\nu\nabla^2)$, but is insensitive to whether or not S is an "active" or passive scalar. To show that S is well mixed within a bounding streamline C_n (see Fig. 1), consider the $t\to\infty$ limit, where

$$\nabla\phi \times \hat{\mathbf{z}} \cdot \nabla S = \nabla \cdot (\nu \nabla S). \tag{B2}$$

Then, integrating Eq. (B2) over the enclosed area gives

$$\int_{A_n} d^2x \mathbf{v} \cdot \nabla S = \int_{A_n} d^2x \nabla \cdot (\nu \nabla S). \tag{B3}$$

However, since $\nabla \cdot \mathbf{v}=0$, Gauss theorem gives

$$\int_{A_n} d^2x \mathbf{v} \cdot \nabla S = \int_{C_n} d\ell\, \hat{\mathbf{n}} \cdot \mathbf{v} S \tag{B4}$$

where $\hat{\mathbf{n}}$ is a unit vector normal to the bounding streamline and $d\ell$ is the differential increment along the streamline. Since $\hat{\mathbf{n}}$ is normal to the streamline C_n, we have $\hat{\mathbf{n}} \cdot \mathbf{v}=0$, so it follows that

$$\int_{A_n} d^2x \nabla \cdot \mathbf{v} \nabla S = \int_{C_n} d\ell\, \hat{\mathbf{n}} \cdot \mathbf{v} \nabla S_n = 0. \tag{B5}$$

Here, S_n is the value of S along the streamline, which since $t \to \infty$, must be an isopotential. Thus, $S_n = S_n(\phi)$. Hence,

$$\int_{C_n} d\ell\, \hat{\mathbf{n}} \cdot \nabla\phi \nu \frac{\delta S_n}{\delta\phi} = 0. \tag{B6}$$

Note, however, that

$$\int_{C_n} d\ell\, \hat{\mathbf{n}} \cdot \nabla\phi = \int_{C_n} d\mathbf{l} \cdot \nabla\phi \times \hat{z} = \gamma_n,$$

where γ_n is the circulation around the contour C_n. Thus, we arrive at

$$\nu \frac{\delta S_n}{\delta\phi} \gamma_n = 0,$$

so $\delta S_n/\delta\phi=0$ necessary. Since C_n is not special, *any* interior contour is equivalent, so $\delta S_n/\delta\phi=0$ for all n, so $\delta S/\delta\phi=0$. Therefore, there is no variation from streamline to streamline within the outermost closed contour C_0, *so S is homogenized within* C_0. In short, then

$$\nabla S = 0$$

within C_0, so S is mixed (homogenized), and ∇S is relaxed.

Several comments are in order here. First, the essential elements of the argument above are that $\nabla \cdot \mathbf{v}=0$, so that $(d/dt)S=0$, up to only diffusive dissipation. Second, it does not matter whether S is an active scalar (as in vorticity or potential vorticity) or a passive scalar. Third, the nature of C_0 and ν is flexible. In this regard, C_0 can be exact, so that ν corresponds to the molecular diffusivity, or C_0 can be approximate, i.e., coarse-grained, where $\nu=\nu_T$, a turbulent diffusivity which includes effects from fluctuations on scales smaller than that of the coarse graining. In particular, C_0 can be a closed streamline bounding the system, so that, given fluid excitation, mixing will continue until $\nabla S=0$ throughout. Finally, the time scale of homogenization is not specified, but will be determined by both diffusion and the time scale for shearing by bounded, circulating flow.

For transport problems in magnetic fusion energy, the programmatic "bottom line" of homogenization theory is that a scalar field which is advected by "incompressible turbulent flow" will be homogenized, so that only the gradient in the mean of that scalar will relax and flatten. Thus, homogenization implies that the flux of the mean S, denoted by $\langle S\rangle$, may be written as

$$\mathbf{\Gamma}_S = -D_S \nabla \langle S\rangle,$$

where D_S is fluctuation-driven, and usually at least estimated by some sort of quasilinear closure, sometimes with renormalization. In confinement devices, mean quantities are functions of the flux surface, so

$$\Gamma_S = -D_S \frac{\partial}{\partial r}\langle S\rangle.$$

Now, throughout the above discussion, we have assumed S to be a single quantity and the advective flow to be incompressible. In a sense, all that TEP theory involves is the possibility that compressibility of the advecting flow results in a situation where a *ratio* or *product* of two fields is effectively advected. A particularly simple example[58] is that of 2D $\mathbf{E}\times\mathbf{B}$ mixing of density in an inhomogeneous, but straight, magnetic field i.e., $\mathbf{B}=B(x,y)\hat{\mathbf{z}}$. Then, from the continuity equation and $\mathbf{E}\times\mathbf{B}$ flow, we have

$$\frac{\partial}{\partial t} n + \nabla \cdot (n\mathbf{v}) = \nu \nabla^2 n, \tag{B7}$$

with

$$\mathbf{v} = -\frac{c}{B} \nabla\phi \times \hat{\mathbf{z}}.$$

We find a rescaled version of the density evolution equation is just

$$\frac{\partial}{\partial t} n + c\nabla\phi \times \hat{\mathbf{z}} \cdot \nabla\left(\frac{n}{B}\right) = \nu \nabla^2 n. \tag{B8}$$

Note that this equation almost has the form of Eq. (B1), with $S=n$, except that the ratio n/B, not n, is advected, on account of the compressibility of the $\mathbf{E}\times\mathbf{B}$ flow induced by the inhomogeneity of B. Thus, n/B is locally conserved up to dissipation of n. Now, it is important to note that $c\int d\ell \nabla\phi \times \hat{\mathbf{z}} \cdot \nabla(n/B)=0$ here, so that homogenization will still occur. However, homogenization theory would then immediately predict that the spatial profile of the mean n/B would relax according to

$$\Gamma_{n/B} = -D_{n/B} \frac{\partial}{\partial x}\langle n/B\rangle,$$

so that

$$\frac{\partial}{\partial t}\langle n\rangle + \frac{\partial}{\partial x}\Gamma_{n/B} = 0. \tag{B9}$$

This, at long last, brings us to TEP theory. Of course, what is of interest is *not* the flux of n/B, but the flux of n, and the evolution of n. Thus, writing out the various terms in Eq. (B9) explicitly gives

$$\frac{\partial}{\partial t}\langle n\rangle = \frac{\partial}{\partial x}\left\{\frac{D_{n/B}}{\bar{B}}\frac{\partial}{\partial x}\langle n\rangle - \frac{\langle n\rangle}{\bar{B}^2}D_{n/B}\frac{\partial}{\partial x}\bar{B}\right\} = \frac{\partial}{\partial x}\left\{D_n\frac{\partial}{\partial x}\langle n\rangle + V_n^p\langle n\rangle\right\}, \quad \text{(B10)}$$

where $\bar{B} \equiv \frac{\langle n\rangle}{\langle n/B\rangle} \simeq \langle B\rangle$,

$$D_n \equiv \frac{D_{n/B}}{\bar{B}},$$

and

$$V_n^p \equiv -\frac{D_{n/B}}{\bar{B}^2}\frac{\partial}{\partial x}\bar{B}.$$

In other words, homogenization and relaxation of gradients of the locally advected quantity, mean n/B, appear as "diffusion and advection" of density. Here, D_n and V_n^p are the diffusion coefficient and pinch velocity, assuming that the n-dependence of $\bar{B}$ is negligible. Note that both quantities have $D_{n/B}$, the original diffusion coefficient for n/B, as a common factor. V_n^p is inward for $(\partial/\partial x)\bar{B}>0$, and constitutes a pinch in that case. Thus, the density profile is stationary for mean profiles which satisfy

$$\frac{\partial}{\partial x}\langle n/B\rangle = 0,$$

or n/B=const in terms of mean values. These are termed "canonical" profiles, and are simply those for which the mean profile of the locally advected quantity is flat. It is interesting to note that the pinch velocity is driven by $\partial B/\partial x$ which is not a thermodynamic force (i.e., not related to a moment of the distribution function). This is not surprising, since the pinch arises from local conservation of n/B, and not from some competition of thermodynamic forces and fluxes, as does a thermoelectric pinch. Finally, we note that: (i) what is ultimately of relevance is the $t\to\infty$ limit of Eq. (B8), and (ii) the scales of B (a mean fixed quantity) are much more slowly varying than n (a local fluctuating quantity), so it is a reasonable approximation to let $n\to n/B$ in the diffusion term on the RHS of Eq. (B8). At that point, homogenization theory applies and the rest follows directly. Section IV in the main text contains an application of the concept of TEP fluxes and homogenization to the momentum transport problem.

APPENDIX C: NONLINEAR GYROKINETIC DERIVATION OF THE LINEAR MOMENTUM FLUX

In Appendix C, we calculate the radial flux of the linear momentum density from the nonlinear gyrokinetic equation, and present fully kinetic expressions. We intentionally consider the linear momentum instead of the angular momentum, to contrast the dependence of the final results on the major radius $R\propto 1/B$. We present a more traditional integration in $(v_\perp, v_\parallel)$-space, rather than in terms of $(\mu, v_\parallel)$. The gyrokinetic equation is[19]

$$\left[\frac{\partial}{\partial t} + \left(v_\parallel\frac{\mathbf{B}^*}{B^*} + \frac{\mu}{e}\frac{\hat{\mathbf{b}}}{B^*}\times\nabla B\right)\cdot\nabla\right]\delta f = -\frac{\hat{\mathbf{b}}}{B^*}\times\nabla\langle\delta\Phi\rangle\cdot\nabla F_0 + \frac{e}{m}\frac{\mathbf{B}^*}{B^*}\cdot\nabla\langle\delta\Phi\rangle\frac{\partial}{\partial v_\parallel}F_0. \quad \text{(C1)}$$

Assuming F_0 is a shifted Maxwellian, we can write

$$\delta f_k = J_0\left(\frac{v_\perp k_\perp}{\Omega}\right)\frac{\left[(v_\parallel - U_0)\frac{\mathbf{B}^*}{B^*}\cdot\mathbf{k} - \frac{T}{e}\frac{\hat{\mathbf{b}}}{B^*}\times\frac{\nabla F_0}{F_0}\cdot\mathbf{k}\right]}{\left[\omega - \left(v_\parallel\frac{\mathbf{B}^*}{B^*} + \frac{\mu}{e}\frac{\hat{\mathbf{b}}}{B^*}\times\nabla B\right)\cdot\mathbf{k}\right]} \times\frac{e}{T}\delta\Phi_k F_0. \quad \text{(C2)}$$

The parallel velocity moment gives the parallel momentum

$$\delta(nv_\parallel)_k = 2\pi\int_{-\infty}^{\infty}dv_\parallel\int_0^{\infty}dv_\perp\left[J_0\left(\frac{k_\perp v_\perp}{\Omega_i}\right)v_\perp v_\parallel\delta f_k\right], \quad \text{(C3)}$$

where we used $Bd\mu\to v_\perp dv_\perp$.

Substituting (C2) into (C3), after some algebra, we obtain

$$\delta(nv_\parallel) = -\frac{n_0(r)}{k_\parallel^*}\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\infty}dx_\parallel\int_0^{\infty}dx_\perp\frac{J_0(\sqrt{2}\rho_i k_\perp x_\perp)^2 x_\perp e^{-x^2}}{(x_\parallel - \zeta_\alpha - \zeta_{D\perp}x_\perp^2/2 - \zeta_{D\parallel}x_\parallel^2)} \times\left\{x_\parallel^2\sqrt{2}v_{ti}k_\parallel^* + x_\parallel\left(k_\parallel U_0 + 2v_{ti}^2x_\parallel^2\frac{\hat{\mathbf{b}}}{\Omega}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{k}\right) - \frac{T}{eB}\left(x_\parallel + \frac{U_0}{\sqrt{2}v_{ti}}\right)L_F^{-1}k_y\right\}, \quad \text{(C4)}$$

where

$$k_\parallel^* \equiv \left[k_\parallel + 2\frac{U_0}{v_{ti}}\rho_i\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{k}\right],$$

$$\zeta_\alpha \equiv \frac{\omega - U_0k_\parallel}{\sqrt{2}v_{ti}k_\parallel^*}, \quad \zeta_{D\parallel} \equiv \frac{2v_{ti}\rho_i\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{k}}{\sqrt{2}v_{ti}k_\parallel^*},$$

$$\zeta_{D\perp} \equiv \frac{2v_{ti}\left(\frac{1}{B}\frac{dB}{dr}\right)\rho_i k_y}{\sqrt{2}v_{ti}k_\parallel^*},$$

$$b \equiv \rho_i^2k_\perp^2, \quad x_\perp \equiv \frac{v_\perp}{\sqrt{2}v_{ti}}, \quad x_\parallel \equiv \frac{(v_\parallel - \bar{v}_\parallel)}{\sqrt{2}v_{ti}},$$

and

$$L_F^{-1} \equiv \left(\frac{1}{\bar{n}}\frac{d\bar{n}}{dr}\left[1-\eta_i\left(\frac{3}{2}-x_\perp^2-x_\parallel^2\right)\right] + \sqrt{2}\frac{x_\parallel}{v_{ti}}\frac{d\bar{v}_\parallel}{dr} - x_\perp^2\frac{1}{B}\frac{dB}{dr}\right).$$

Defining

$$I_{nm} = \frac{2}{\sqrt{\pi}}\int_0^\infty dx_\perp \int_{-\infty}^{\infty} dx_\parallel \frac{x_\perp^n x_\parallel^m J_0^2(\sqrt{2b}x_\perp)}{(x_\parallel-\zeta_\alpha-\zeta_{D\perp}x_\perp^2/2-\zeta_{D\parallel}x_\parallel^2)}e^{-x^2},$$

we can write

$$\delta(nv_\parallel)_k = \frac{n_0(r)}{k_\parallel^*}\frac{e\delta\Phi_k}{T}\left\{\left(\sqrt{2}v_{ti}k_\parallel^* - \frac{U_0\omega_{*i}\eta_i}{\sqrt{2}v_{ti}} + \omega_{*i}\sqrt{2}\frac{L_n}{v_{ti}}\frac{dU_0}{dr}\right)I_{12} + \left[k_\parallel U_0 - \omega_{*i}\left(1-\frac{3}{2}\eta_i\right)\right]I_{11} + \left(2v_{ti}^2\frac{\hat{\mathbf{b}}}{\Omega}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{k} - \omega_{*i}\eta_i\right)I_{13} - \omega_{*i}\left(1-\frac{3}{2}\eta_i\right)\frac{U_0}{\sqrt{2}v_{ti}}I_{10} - \omega_{*i}\left(\eta_i + L_n\frac{1}{B}\frac{dB}{dr}\right)I_{31} - \omega_{*i}\left(\eta_i + L_n\frac{1}{B}\frac{dB}{dr}\right)\frac{U_0}{\sqrt{2}v_{ti}}I_{30}\right\}. \tag{C5}$$

Notice that, according to the usual convention, $\omega_{*i}<0$ for $k_y>0$, and $\omega \propto \omega_{*i}$ for the ITG mode. Note also that the I_{nm}'s themselves are complicated functions of $k_\parallel^*$ and other variables, because of their dependence on the parameters ζ_α, $\zeta_{D\parallel}$ and $\zeta_{D\perp}$. Therefore it is not easy to identify the resulting "net momentum pinch" analytically in the fully kinetic expression. Instead, we take the fluid limit of (C5), as a confirmation of the previous result, using the fluid limit of I_{nm},

$$I_{nm} \sim -\frac{\Gamma\left(\frac{n+1}{2}\right)}{\zeta_\alpha\sqrt{\pi}}\begin{cases}\frac{1}{\zeta_\alpha}\Gamma\left(\frac{m+2}{2}\right) & m\text{:odd}\\ \Gamma\left(\frac{m+1}{2}\right)\left[1+\left(\frac{m+1}{2}\right)\left(\zeta_\alpha^{-2}-\frac{\zeta_{D\parallel}}{\zeta_\alpha}\right)-\left(\frac{n+1}{2}\right)\left(\frac{\zeta_{D\perp}}{2\zeta_\alpha}+b\right)\right] & m\text{:even}\end{cases}.$$

This gives

$$\delta(nv_\parallel)_k = v_{ti}\frac{n_0(r)}{\omega}\frac{e\delta\Phi_k}{T}\left\{v_{ti}\left(1-\frac{\omega_{*i}(1+\eta_i)}{\omega}-\frac{\omega_{*i}}{\omega}\frac{L_n}{B}\frac{dB}{dr}\right)\left[k_\parallel + 2\frac{U_0}{v_{ti}}\rho_i\hat{\mathbf{b}}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{k}\right] + \omega_{*i}\frac{L_n}{v_{ti}}\frac{dU_0}{dr} - \left(1+\frac{L_n}{B}\frac{dB}{dr}\right)\frac{\omega_{*i}U_0}{v_{ti}}\right\} \tag{C6}$$

where the last two terms may be combined as $[\omega_{*i}L_nB/(v_{ti}n_0)]\times(d/dr)(U_0n_0/B)$. It should be understood that, in fluid limit formulas in this Appendix C, $1/\omega$ should be interpreted as $1/(\mathrm{Re}\,\omega+i|\gamma|)$ as required by the causality constraint.

The Reynolds stress is

$$\langle\delta v_{Er}\delta(nv_\parallel)\rangle = \mathrm{Re}\sum_k iv_{ti}\rho_i k_y\frac{n_0(r)}{k_\parallel^*}\left|\frac{e\delta\Phi_k}{T}\right|^2\left\{\left(\sqrt{2}v_{ti}k_\parallel^* - \frac{\omega_{*i}\eta_i U_0}{\sqrt{2}v_{ti}} + \omega_{*i}\sqrt{2}\frac{L_n}{v_{ti}}\frac{dU_0}{dr}\right)I_{12} + \left[k_\parallel U_0 - \omega_{*i}\left(1-\frac{3}{2}\eta_i\right)\right]I_{11} + \left(2v_{ti}^2\frac{\hat{\mathbf{b}}}{\Omega}\times(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{k} - \omega_{*i}\eta_i\right)I_{13} - \omega_{*i}\left(1-\frac{3}{2}\eta_i\right)\frac{U_0}{\sqrt{2}v_{ti}}I_{10} - \omega_{*i}\left(\eta_i + L_n\frac{1}{B}\frac{dB}{dr}\right)I_{31} - \omega_{*i}\left(\eta_i + L_n\frac{1}{B}\frac{dB}{dr}\right)\frac{U_0}{\sqrt{2}v_{ti}}I_{30}\right\}, \tag{C7}$$

which becomes

$$\langle\delta v_{Er}\delta(nv_\parallel)\rangle = \mathrm{Re}\sum_k iv_{ti}^2\frac{\rho_i k_y}{\omega}\left|\frac{e\delta\Phi_k}{T}\right|^2\left\{v_{ti}n_0(r)\left(1-\frac{\omega_{*i}(1+\eta_i)}{\omega}-\frac{\omega_{*i}}{\omega}\frac{L_n}{B}\frac{dB}{dr}\right)\times\left[k_\parallel + 2\frac{U_0}{v_{ti}}\rho_i\hat{\mathbf{b}}\times\left(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}}\right)\cdot\mathbf{k}\right] + \omega_{*i}\frac{L_nB}{v_{ti}}\frac{d}{dr}\left(\frac{n_0U_0}{B}\right)\right\} \tag{C8}$$

when we take the fluid limit. Note that Eq. (C8) exhibits the $\mathbf{B}^*$-symmetry breaking mechanism via $\left[k_\parallel + 2\frac{U_0}{v_{ti}}\rho_i \hat{\mathbf{b}} \times (\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{k}\right]$. We can also see that the result for $V^{\mathrm{TEP}}_{\mathrm{Mom}}$ in Eq. (10) is fully recovered. Terms related to $\eta_i\omega_{*i}$ correspond to $V^{\mathrm{CTh}}_{\mathrm{Mom}}$. Noting that $\delta T_i/T_i \sim \eta_i(\omega_{*i}/\omega)e\delta\phi/T_e$, we recover part of the result for $V^{\mathrm{CTh}}_{\mathrm{Mom}}$ in Eq. (11). The slight difference in coefficients is due to the fact that different versions of the propagators have been used in taking the fluid limit.

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